MATH 8320: ALGEBRAIC CURVES AND RIEMANN SURFACES – HOMEWORK 5

Throughout, X is a compact Riemann surface of genus g. By default, divisors, meromorphic functions, et cetera are on X.

1. Ample divisors

(1) Recall that a very ample divisor is the hyperplane divisor under a closed embedding. Show that a very ample divisor is ample.

(2) Show that A is ample if and only if some multiple nA for $n > 0$ is very ample.

(3) Use Riemann–Roch to show that any divisor of positive degree is ample.

Hint: Feel free to use the existence of an ample divisor. Also remember that A is ample if and only if nA is ample, and if A is ample and E is effective, then $A + E$ is ample.

2. Serre duality

(4) Let $p \in X$, and t a uniformizer at p. Let

$$\alpha(t) = \sum_{i=1}^{n} a_i t^{-i}$$

interpreted as an element of $\mathbb{C}(\!(t)\!)/\mathbb{C}[\![t]\!]$. Show that Serre duality says the following: There exists a meromorphic function on X, holomorphic away from p, with Laurent tail $\alpha(t)$ if and only if the coefficients a_i satisfy certain g linear conditions.

(5) Explicitly write down the g linear conditions when (X, p) are as follows:
   (a) X is $y^2 = x^6 - 1$ (compactified), $p = (0, i)$, and $t = x$.
   (b) X is $y^2 = x^6 - 1$ (compactified), $p = (1, 0)$, and $t = y$.

3. Vanishing sequences

(6) Let L be a line bundle. The vanishing sequences in this problem are with respect to the complete linear series $(L, H^0(X, L))$. Let $r = h^0(X, L)$. Fix a point $p \in X$ and consider the function $\tau\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ defined by

$$\tau(n) = h^0(X, L(-np)).$$

   (a) Show that $\tau(n) - 1 \leq \tau(n+1) \leq \tau(n)$ and $\tau(n) = 0$ for $n > \deg L$.
   (b) Show that the vanishing sequence of p consists of exactly those n where τ drops; that is, where $\tau(n) = \tau(n-1) - 1$.

(7) The *canonical* vanishing sequence is the vanishing sequence with respect to the canonical series. Show that the canonical vanishing sequence at p is given by

$$\left\{n \in \mathbb{Z}_{\geq 0} \mid h^1(X, np) = h^1(X, (n-1)p).\right\}$$

4. Weierstrass points

(8) Let $g \geq 2$. Let X be hyperelliptic and $\phi\colon X \to \mathbb{P}^1$ the unique degree 2 map. Show that the Weierstrass points are precisely the ramification points of ϕ.

(9) Let X be hyperelliptic. Write down the canonical vanishing sequence at a Weierstrass point of X and a non-Weierstrass point of X. What is the multiplicity of the Wronskian at the Weierstrass point?

(10) Show that for the canonical series, the highest order of vanishing of the Wronskian at p can be $g(g-1)/2$, and equality holds if and only if X is hyperelliptic and $p \in X$ is a Weierstrass point. Conclude that on X, there are at least $2g+2$ (distinct) Weierstrass points.

(11) Figure out the connection between problem (4) and the canonical vanishing sequence.