ALGEBRAIC GEOMETRY: PRACTICE QUESTIONS

You should be able to do all the questions on all the homeworks and all the workshop questions. In addition, here are more practice questions.

(1) Let $X \subset \mathbb{A}^3$ be the union of the three coordinate axes. Find $I(X)$.
(2) Let $I, J \subset k[x_1, \ldots, x_n]$ be ideals. Denote byIJ the ideal generated by $\{fg \mid f \in I, g \in J\}$. Prove that $\sqrt{IJ} = \sqrt{I \cap J}$.
(3) Describe explicitly all the maximal ideals of the ring $k[x, y]/(x^2 - y^3)$.
(4) Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^n$ be disjoint Zariski closed sets. Show that there exists a polynomial $f \in k[x_1, \ldots, x_n]$ such that the function $f|_X$ is identically 0 and $f|_Y$ is identically 1.
(5) Give an example of a regular map $f\colon X \to Y$ that is a bijection, but the inverse map is not regular.
(6) $\operatorname{char} k \neq 2$. Construct an isomorphism between the affine variety defined by $x^2 + y^2 = 1$ in $\mathbb{A}^2$ and the affine variety defined by $xy = 1$ in $\mathbb{A}^2$.
(7) Construct an isomorphism from $\mathbb{A}^1 \setminus \{0, 1\}$ to a Zariski closed subspace of $\mathbb{A}^n$ for some n.
(8) As usual, think of $\mathbb{A}^3 \subset \mathbb{P}^3$ as the open subset where the last homogeneous coordinate is nonzero. Find the closure in $\mathbb{P}^3$ of $V(x - y^4, x - z^7)$.
(9) Let m, n be positive integers, and let $f_1, \ldots, f_n \in k[x_1, \ldots, x_m]$. Consider the map $f\colon \mathbb{A}^m \to \mathbb{A}^n$ defined by

$$f(p) = (f_1(p), \ldots, f_n(p)).$$

Show that f is surjective if and only if for all $a_1, \ldots, a_n \in k$, the ideal $\langle f_1 - a_1, \ldots, f_n - a_n \rangle$ is not the unit ideal.
(10) Let $U \subset \mathbb{P}^n$ be a quasi projective variety. When is a map $f : U \to \mathbb{P}^m$ called a regular map?
(11) Let $X = V(y^2 - x^3)$ and $U = X \setminus \{(0, 0)\}$. Show that the function y/x on U does not extend to a regular function on X.
(12) Prove that every regular map $\mathbb{P}^1 \to \mathbb{A}^1$ is a constant.
(13) Prove that the hyperbola, defined by $xy = 1$ in $\mathbb{A}^2$, and the line $\mathbb{A}^1$ are not isomorphic.
(14) Prove that all isomorphisms $f\colon \mathbb{A}^1 \to \mathbb{A}^1$ are of the form $f(x) = ax + b$.
(15) Prove that any two non-empty Zariski open subsets of $\mathbb{P}^n$ have a non-empty intersection.
(16) Prove that every affine variety is (isomorphic to) a quasi-projective variety.

(17) Let $v\colon \mathbb{P}^2 \longrightarrow \mathbb{P}^5$ be the degree 2 Veronese map. Describe the image under v of the line $Z = 0$ in $\mathbb{P}^2$.

(18) Let $v_m\colon \mathbb{P}^n \longrightarrow \mathbb{P}^N$ be the degree m Veronese map. Prove that the linear span of $v_m(\mathbb{P}^n)$ is $\mathbb{P}^N$.

(19) char $k \neq 2$. Construct an isomorphism from the quadric surface $X^2 + Y^2 + Z^2 + W^2 = 0$ in $\mathbb{P}^3$ to $\mathbb{P}^1 \times \mathbb{P}^1$.

(20) Let $C \subset \mathbb{P}^2$ be the Fermat cubic curve
$$C = V(X^3 + Y^3 + Z^3).$$
   (a) Show that the linear projection $[X : Y : Z] \mapsto [X+Y : Z]$ when restricted to C extends to a regular map $\phi\colon C \longrightarrow \mathbb{P}^1$.
   (b) Describe the fibers of ϕ.