Regular functions and regular maps.

k = Alg. closed field.
Recall from last time:
$X \subset \mathbb{A}^n_k$ affine algebraic set.
$f: X \to k$ <u>regular</u> if it is the restriction of a polynomial function.

$$\begin{aligned} k[X] &= k\text{-algebra of regular functions on } X \\ &\cong k[x_1, \dots, x_n] / I(X). \\ &= \text{Finitely generated nilpotent free } k\text{-algebra.} \end{aligned}$$

<u>Observe</u> - Any finitely generated nilpotent free k-algebra is of the form $k[X]$ for some X.

Why? Let A be such an algebra.
Let $a_1, \dots, a_n \in A$ be a set of generators.
Then we have a map

$$\varphi: k[x_1, \dots, x_n] \to A$$
$$x_i \mapsto a_i.$$

This map is surjective because $\{a_i\}$ generates A. By the first iso thm

$$A \cong k[x_1, \dots, x_n] / I$$

where $I = \operatorname{Ker} \varphi$.
Since A is nilpotent free, I is radical.
Then take $X = V(I)$.
By the Nullstellensatz,

$$\begin{aligned} k[X] &= k[x_1, \dots, x_n] / I(X) \\ &= k[x_1, \dots, x_n] / I \\ &\cong A \end{aligned}$$

$\square$

As a result we have the dictionary.

Algebra	Geometry
• Finitely generated reduced k-alg. A	• Alg of regular functions on affine alg set X.
• Max ideal of A	• Point of X
• Given $J \subset A$. $V(J) = \{ m \mid m \supset J \}$.	• Given $J \subset k[X]$ $V(J) = \{ x \mid f(x) = 0 \ \forall f \in J \}$

In particular $V(J) = \phi$ iff $J = (1)$.

Regular Maps

$X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ affine alg sets.
$f: X \to Y$ is a <u>regular</u> function if

$\exists\ f_1, \dots, f_m \in k[X]$ such that

$$f(x) = (f_1(x), \dots, f_m(x)) \quad \forall\ x \in X.$$

Equivalently, if there exist $F_1, \dots, F_m$ in $k[X_1, \dots, X_n]$ such that
$$f(x) = (F_1(x), \dots, F_m(x)) \quad \forall\ x \in X.$$

<u>Ex 1</u>: $f: X \to \mathbb{A}^1$ regular map
$\iff$ f is a regular function.

<u>Ex 2</u>: $L: \mathbb{A}^n \to \mathbb{A}^m$ linear transf^n is regular.

<u>Ex 3</u>: Projections $\mathbb{A}^n \to \mathbb{A}^1$

<u>Ex 4</u>: Compositions of regular maps are regular

Ex 5: $X \subset \mathbb{A}^n$ Zariski closed.
The inclusion $X \to \mathbb{A}^n$ is regular.

Def: A regular $f: X \to Y$ is an isomorphism if there exists a regular inverse map $g: Y \to X$.

Ex 6: $X = \mathbb{A}^1$
$Y = \{y^2 - x^3 = 0\} \subset \mathbb{A}^2$

$$f: X \to Y$$
$$t \mapsto (t^2, t^3)$$
is a regular bijection but not an isomorphism!
How does one see that it's not an iso? Wait and see....

Let $\varphi: X \to Y$ be any map.
Then we get an induced map

$$\varphi^*: \text{Functions on } Y \to \text{Functions on } X$$
$$f \mapsto f \circ \varphi.$$

Proposition: φ is regular if and only if φ^* sends regular functions on Y to regular functions on X.

Pf: Suppose φ is regular.
If $f: Y \to \mathbb{A}^1$ is a regular function then $\varphi \circ f$ is regular because composition of regular maps is regular.
Conversely, suppose $\varphi^*(f)$ is regular for every regular f. Let $\varphi(x) = (\varphi_1(x), \dots, \varphi_m(x))$
We want to show each $\varphi_i(x)$ is regular
But $\varphi_i = \varphi^*(x_i)$ and $x_i \in k[Y]$ is regular. $\square$

Thus a regular map $\varphi: Y \to X$ induces a k-alg. hom $\varphi^*: k[Y] \to k[X]$.

Prop: Let $\alpha: K[Y] \to K[X]$ be a k-alg hom. Then there is a unique regular $\varphi: X \to Y$ such that $\alpha = \varphi^*$.

Pf: suppose $Y = V(J) \subset \mathbb{A}^m$
and $X = V(I) \subset \mathbb{A}^n$.

Then $k[Y] = k[y_1, \dots, y_m] / J$

$k[X] = k[x_1, \dots, x_n] / I$.

Let $\varphi_i = \alpha(y_i) \in k[X]$

Consider $\varphi := (\varphi_1, \dots, \varphi_m) : X \to \mathbb{A}^m$.

Let us check that φ maps X to Y.

To see this, we must show that

$$f(\varphi_1(x), \dots, \varphi_m(x)) = 0 \quad \forall \; x \in X, \; f \in J.$$

But $f(\varphi_1(x), \dots, \varphi_m(x))$

$$= f(\alpha(y_1), \dots, \alpha(y_m))$$

$$= \alpha \, f(y_1, \dots, y_m)$$

$$= \alpha(0) = 0.$$

So $\varphi : X \to Y$. Note $\varphi^*(y_i) = \alpha(y_i)$

so $\varphi^* = \alpha$ because $\{y_i\}$ generate $k[Y]$.

Finally, if $\varphi : X \to Y$ is such that $\varphi^* = \alpha$, and $\varphi = (\varphi_1, \dots, \varphi_m)$, then $\varphi^*(y_i) = \varphi_i = \alpha(y_i)$, so there is only one possible φ.

$\square$.

Conseq: $X \rightsquigarrow k[X]$ defines an equivalence of categories

$$\left\{ \begin{array}{l} \text{Affine alg} \\ \text{sets with} \\ \text{regular maps} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{Fin gen reduced} \\ k\text{-algebras} \\ \text{with } k\text{-alg.} \\ \text{homs} \end{array} \right\}$$

Ex: $X = \mathbb{A}^1$
$Y = V(y^2 - x^3) \subset \mathbb{A}^2$

$k[X] = k[t] \qquad k[Y] = k[x,y] / (y^2 - x^3)$

$\varphi : X \to Y \qquad \varphi(t) = (t^2, t^3)$

$$\varphi^* : k[Y] \to k[X]$$
$$x \mapsto t^2$$
$$y \mapsto t^3.$$

φ^* is not an isomorphism!
Any element in the image of φ^* has vanishing linear term.

Def : Affine algebraic variety
$=$ Affine algebraic set.

We eventually want to define more general algebraic varieties. The first step is

Def : Quasi-affine varieties = Zariski open subsets of affine alg. var.

We now define regular functions and regular maps for quasi-affines.

Def : $U \subset X$ open.
$f : U \to k$ regular if the following holds – $\forall\ x \in U$ there exists an open U_x containing x & $F_x, G_x \in k[X]$ such that G_x is nowhere 0 on U_x and
$$f = F_x / G_x \text{ on } U_x.$$

Example ①: $U = \mathbb{A}^1 - \{0\} \subset \mathbb{A}^1$.
Then $\frac{1}{t}$ is regular on U.

(2) $X = \{xy - z^2 = 0\} \subset \mathbb{A}^3$
$U = X \setminus \{(x,0,0) \mid x \in k\}$.

$f = \dfrac{x}{z}$ or $\dfrac{z}{y}$ is regular on U.

Before we proceed, we must show that we get the same notion of regular as before for affines.

Prop: Let $X \subset \mathbb{A}^n$ be Zar. closed. $f: X \to k$ is regular in the new sense (locally poly/poly) iff it is regular in the old sense (globally a polynomial).

Pf: Let $x \in X$. There exist U_x, F_x, G_x such that $f = F_x / G_x$ on U_x & $x \in U_x$. Say $U_x = X \setminus V(I_x)$. Take $H_x \in I_x$ such that $H_x(x) \neq 0$. Replace U_x by $U'_x = X \setminus V(H_x) \subset U_x$.

F_x by $A_x = F_x H_x$ and

G_x by $B_x = G_x H_x$.

Then $f = \frac{A_x}{B_x}$ on U'_x,

$x \in U'_x$ and $A_x, B_x = 0$ on the complement of U'_x.

Now $\{B_x \mid x \in X\}$ have no common zero, so by the Nullstellensatz they generate the unit ideal of $k[X]$.
Write

$$1 = C_1 B_{x_1} + \cdots + C_\ell B_{x_\ell}$$

where $C_i \in k[X]$.

Multiply both sides by f

$$f = \sum C_i B_{x_i} \cdot f$$

& note

$$B_{x_i} f = A_{x_i} \quad \text{on } X$$

so $f = \sum C_i A_{x_i} \in k[X]$

$\square$.

Having defined regular functions, we can define regular maps just as before.

Def: $U \subset \mathbb{A}^n$ $V \subset \mathbb{A}^m$ opens in closed.

$\varphi : U \to V$ regular map

$\varphi = (\varphi_1, \ldots, \varphi_m)$ where φ_i is reg. fun.

Obs : ① Pull backs of reg. fun under reg maps are regular
② Compositions of reg. fun are regular

Example (Important).

$X = \mathbb{A}^1 \setminus \{0\}$.
$Y = V(xy-1) \subset \mathbb{A}^2$.

$\varphi : Y \to X$ regular
$(x,y) \mapsto x$.

$\psi : X \to Y$
$x \mapsto (x, \frac{1}{x})$ regular.

$\varphi \circ \psi = id$, $\psi \circ \varphi = id$. So $X \cong Y$.

That is the quasi-affine X is actually affine !.
Ring of reg. fun on $X =$
$k[x,y]/(xy-1) \cong k[t, t^{-1}] \subset k(t)$

by $x \mapsto t$, $y \mapsto t^{-1}$.

<u>Example</u> (Important)

$$X = \mathbb{A}^n - V(f).$$
$$Y = V(yf-1) \subset \mathbb{A}^{n+1}.$$

$\varphi: Y \to X$ regular
$(x,y) \mapsto x$

$\psi: X \to Y$
$x \mapsto \left(x, \frac{1}{f(x)}\right)$ regular.

$\varphi \circ \psi = id$ $\psi \circ \varphi = id$.
so $X \cong Y$. & hence

$$k[X] \cong k[x_1, \dots, x_n, y] / (f(x)y - 1)$$

$$\cong \left\{ \frac{p}{f^m} \;\middle|\; p \in k[x_1, \dots, x_n],\ m \geq 0 \right\}$$

$$\subset k(x_1, \dots, x_n, y)$$

by the map $x_i \mapsto x_i$ $y \mapsto \frac{1}{f}$.

A non-affine variety

$$X = \mathbb{A}^2 - \{0\}$$

We have a map

$$k[X] \longrightarrow k(x,y)$$

$$f \longmapsto \frac{F}{G} \quad \text{where } f = \frac{F}{G} \text{ on}$$

some open U in X.

The choice of U does not matter —
First any two opens in X intersect
$\Rightarrow$ any open is dense.

So if $f = \frac{F_1}{G_1}$ on U_1

$$= \frac{F_2}{G_2} \text{ on } U_2$$

then $G_2 F_1 - F_1 G_2 = 0$ on $U_1 \cap U_2$

$= 0$ on $\mathbb{A}^2$

by continuity. So $F_1/G_1 = F_2/G_2$ in $k(x,y)$.

Write $X = \mathbb{A}^2 \setminus V(x) \cup \mathbb{A}^2 \setminus V(y)$

Now the reg. fun on $\mathbb{A}^2 \setminus V(x)$ in $k(x,y)$ are $\left\{ \frac{f}{x^n} \right\}$

Similarly reg. fun on $\mathbb{A}^2 \setminus V(y)$ are

$\left\{ \frac{f}{y^n} \right\}$.

A reg fun on X must lie in the intersection but the intersection

$$\left\{ \frac{f}{x^n} \,\middle|\, f \in k[x,y] \right\} \cap \left\{ \frac{f}{y^m} \,\middle|\, f \in k[x,y] \right\}$$
$$\parallel$$
$$k[x,y].$$

So $k[X] = k[x,y] = k[\mathbb{A}^2]$.

To conclude that X is not affine see that the ideal $(x,y) \subset k[X]$ is non unit but $V(x,y) = \emptyset$ in X. This does not happen for affine X

$\square$.
