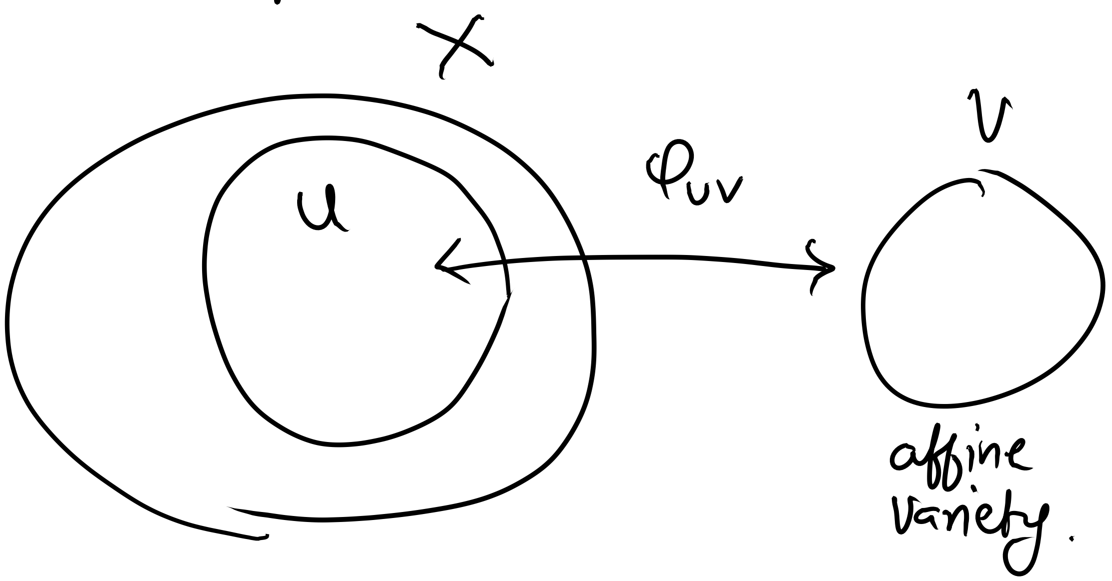
What is an algebraic variety?

The definition of an algebraic variety is very similar to the definition of a manifold in differential geometry.

Def: An algebraic variety is a topological space with an affine atlas.

Affine atlas for X = Collection of compatible collection of affine charts that cover X.

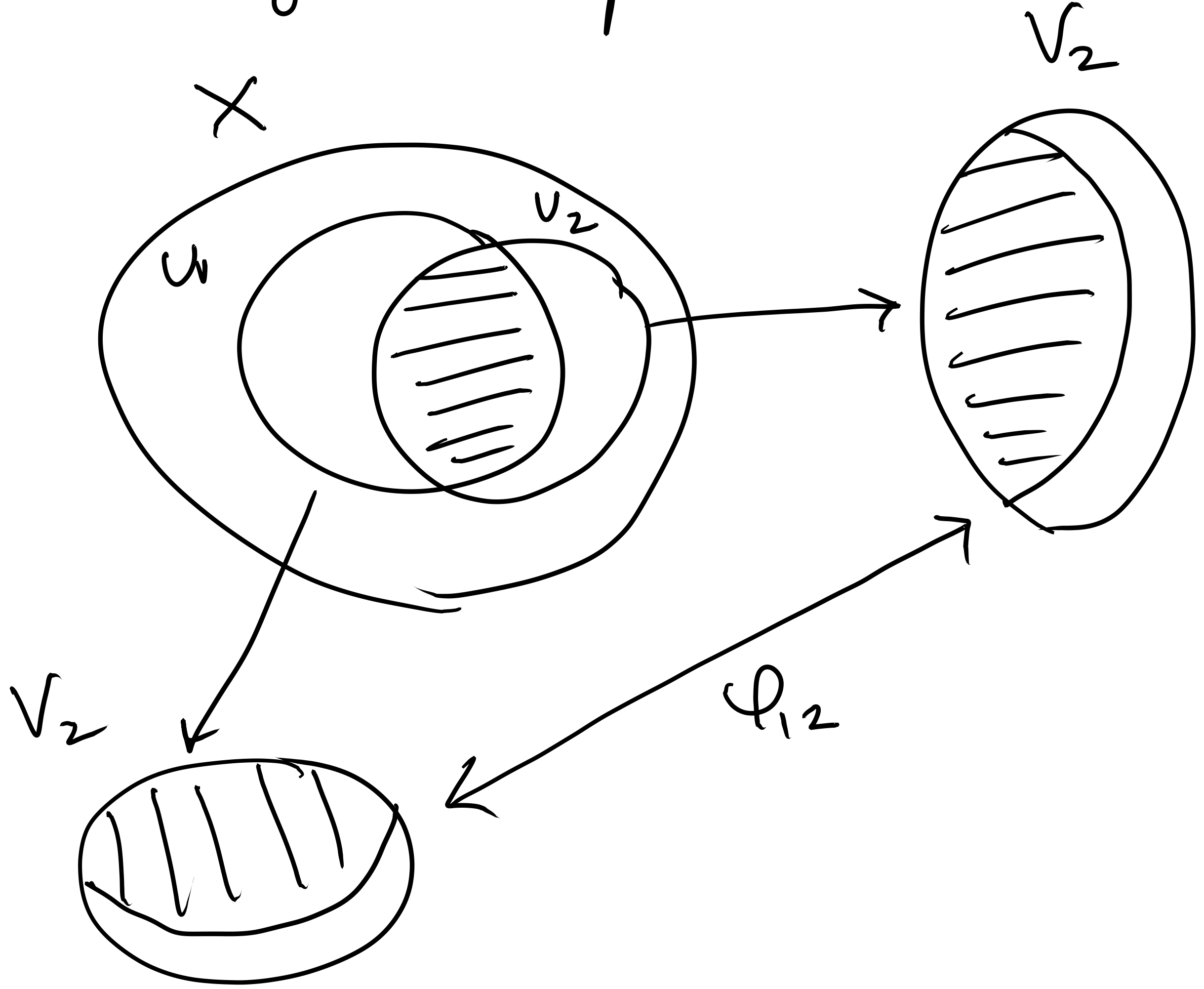
Affine chart : (U, V, φ_{UV})
where $U \subset X$ is open, V is an affine variety and $\varphi_{UV} : U \to V$ is a homeomorphism.



Two charts (U_1, V_1, φ_1), (U_2, V_2, φ_2) are compatible if the map φ_{12}

$$\begin{array}{ccc} \varphi_1(U_1 \cap U_2) & \xrightarrow{\varphi_{12}} & \varphi_2(U_1 \cap U_2) \\ \cap & & \cap \\ V_1 & & V_2 \end{array}$$

is a regular map.



An atlas is a collection of compatible charts $\{(U_i, V_i, \varphi_i)\}$ such that the open sets $\{U_i\}$ cover X.

Example: Quasi affine variety.

$$U = X \setminus V(I), \quad X \text{ affine}, \quad I \subset k[X]$$
$$= \bigcup_{f \in I} X_f, \quad \text{where}$$

$$X_f = \{x \in X \mid f(x) \neq 0\} \leftarrow \text{Affine!}$$

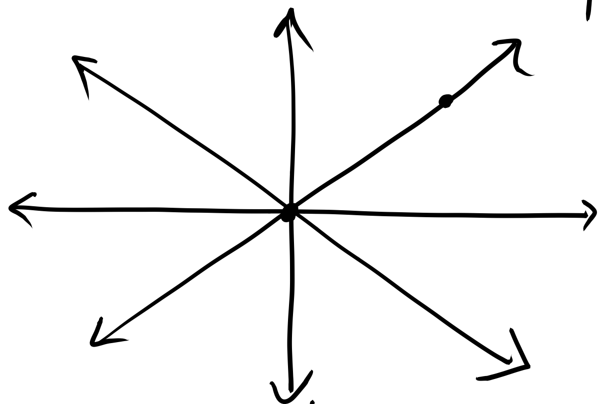
So can take $\varphi_f : X_f \xrightarrow{id} X_f$ & then all the transition maps are just the identity.

Example (Important)

THE PROJECTIVE SPACE.

$$\mathbb{P}^n = \text{Set of lines in } k^{n+1}$$
$$\qquad\qquad\qquad \text{one dim. subspaces}$$

e.g. $\mathbb{P}^1$



$$\mathbb{P}^n = \{(a_0, \dots, a_n) \mid a_i \in k, \text{ not all } 0\}/\sim$$

$$(a_0, \dots, a_n) \sim (\lambda a_0, \dots, \lambda a_n) \quad \lambda \in k^\times$$

Topology: Closed sets of $\mathbb{P}^n$
= Subsets defined by collections of homogeneous polynomials in $k[X_0, \dots, X_n]$

$p(X_0, \dots, X_n)$ is homogeneous of degree

$$p(X_0, \dots, X_n) = \sum_I a_I X^I \quad \text{where } |I| = d$$
$$= \sum_{(i_0, \dots, i_n)} a_{i_0 \cdots i_n} X_0^{i_0} \cdots X_n^{i_n} \quad \Sigma i_j = d.$$

equivalently, if $p(\lambda X_0, \dots, \lambda X_n)$
$= \lambda^d p(X_0, \dots, X_n)$.

Note: Even if p is homogeneous of deg $d > 0$, p DOES NOT define a function on $\mathbb{P}^n$ but it DOES define a "vanishing set"

$$V(p) = \{[x] \in \mathbb{P}^n \mid p(x) = 0\}$$

The equality $p(x) = 0$ does not depend on the chosen representative of $[x]$.
Since
$$p(\lambda x) = \lambda^d p(x),$$
both $p(\lambda x)$ & $p(x)$ are simultaneously zero or non-zero.

Affine charts: (see example of $\mathbb{P}^2$ two pages ahead.)

$$\mathbb{P}^n = \{ [x_0 : \cdots : x_n] \mid \text{Not all } x_i = 0 \}$$
$$= U_0 \cup U_1 \cup \cdots \cup U_n$$

$$U_i = \{ [X_0 : \cdots : X_n] \mid X_i \neq 0 \}$$
$$= \{ [x_0 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n] \mid x_i \in k \}$$
$$\xrightarrow{\varphi_i} \mathbb{A}^n \quad \text{bijection}$$

$$\varphi_i : [X_0 : \cdots : X_n] \mapsto \left(\frac{X_0}{X_i}, \ldots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \ldots, \frac{X_n}{X_i} \right)$$

Claim: φ_i is a homeomorphism.

Pf: Let $Z \subset U_i$ be closed. Then $Z = V(S) \cap U_i$ where S is a set of homogeneous polynomials in $X_0, \ldots, X_n$

Then $\varphi_i(Z) =$
$$V(\{ p(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots x_n) \mid p \in S \}). \subset \mathbb{A}^n.$$

closed.

$$\mathbb{A}^n = \{ (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \}$$

Conversely, consider

$$Y = V(T) \subset \mathbb{A}^n$$

where $T \subset k[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$.

We must construct a set T' of homogeneous polynomials in $X_0, \dots, X_n$ such that

$$\varphi_i^{-1}(Y) = V(T') \cap U_i.$$

T' is obtained from T by homogenizing wrt X_i. For $p \in T$, create

$$p^{hom} = X_i^{\deg p}\, p\left(\frac{X_1}{X_i}, \dots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \dots, \frac{X_n}{X_i}\right)$$

(e.g. $p = x_0^2 + x_1$
$p^{hom} = X_0^2 + X_1 X_i$)

Now set $T' = \{ p^{hom} \mid p \in T \}$.

Then $\varphi^{-1}(V(T)) = V(T') \cap U_i$,
so $\varphi^{-1}(V(T))$ is closed.

Transition functions.

$$
\begin{tikzcd}[column sep=small]
 & & \mathbb{P}^n & & \\
U_i \arrow[urr, phantom, "\subset"] \arrow[d, bend right] & & U_i \cap U_j \arrow[ll, phantom, "\supset"] \arrow[rr, phantom, "\subset"] \arrow[dll] \arrow[drr] & & U_j \arrow[ull, phantom, "\supset"] \arrow[d, bend left] \\
\mathbb{A}^n \cup \{x_j \neq 0\} \arrow[rrrr] & & & & \{x_i \neq 0\} \cup \mathbb{A}^n
\end{tikzcd}
$$

$U_i = \{X_i \neq 0\}$, $[X_0 : \cdots : X_n] \mapsto \left(\frac{X_0}{X_i}, \ldots, \frac{X_n}{X_i}\right)$

$U_i \cap U_j = \{X_i \neq 0 \ \& \ X_j \neq 0\}$

$U_j = \{X_j \neq 0\}$, $[X_0 : \cdots : X_n] \mapsto \left(\frac{X_0}{X_j}, \ldots, \frac{X_n}{X_j}\right)$

$$
\begin{tikzcd}
(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \arrow[r, "\text{regular}"] \arrow[d, mapsto, "\varphi_i^{-1}"] & \left(\frac{x_0}{x_j}, \ldots, \frac{1}{x_j}, \ldots, \widehat{\frac{x_j}{x_j}}, \ldots, \frac{x_n}{x_j}\right) \\
{[X_0 : \cdots : X_{i-1} : 1 : X_{i+1} : \cdots : X_n]} \arrow[ur, mapsto, "\varphi_j"'] &
\end{tikzcd}
$$

($\wedge$ symbol = exclude this coordinate)

Example of $\mathbb{P}^2$

$$\mathbb{P}^2 = \{[X_0 : X_1 : X_2] \mid \text{Not all } X_i = 0\}$$

$$U_0 = \mathbb{P}^2 \setminus V(X_0)$$
$$= \{[1 : X_1 : X_2] \mid X_1, X_2 \in k\}$$

$$\begin{array}{l} U_0 \\ \downarrow \varphi_0 \\ \mathbb{A}^2 \end{array}$$

$$\varphi_0([1 : X_1 : X_2]) = (X_1, X_2)$$
$$\varphi_0([X_0 : X_1 : X_2]) = \left(\frac{X_1}{X_0}, \frac{X_2}{X_0}\right).$$

φ_0 takes closed sets to closed sets.

Example: Take $Z = V(X_0^2 - X_1X_2,\ X_0^3 - X_2^3) \cap U_0$

Then
$$\varphi_0(Z) = V(1 - x_1x_2,\ 1 - x_2^3) \subset \mathbb{A}^2.$$

φ_0^{-1} takes closed sets to closed sets.

Example: Take $Y = V(1 - x_1,\ x_2^2 - x_1^3) \subset \mathbb{A}^2$

Then
$$\varphi_0^{-1}(Y) = U_0 \cap V(X_0 - X_1,\ X_0X_2^2 - X_1^3).$$

Example of transition functions

$$U_0 = \{ [X_0 : X_1 : X_2] \mid X_0 \neq 0 \}$$
$$U_1 = \{ [X_0 : X_1 : X_2] \mid X_1 \neq 0 \}$$

$$U_0 \cap U_1 = \{ [X_0 : X_1 : X_2] \mid X_0 \neq 0 \text{ and } X_1 \neq 0 \}.$$

$$\begin{tikzcd}
U_0 \arrow[rr, phantom, "\supset"] \arrow[d, "\varphi_0"'] & & U_0 \cap U_1 \arrow[rr, phantom, "\subset"] \arrow[dl, "\varphi_0"'] \arrow[dr, "\varphi_1"] & & U_1 \arrow[d, "\varphi_1"] \\
\mathbb{A}^2 \arrow[r, phantom, "\supset"] \arrow[d, equal] & \{x_1 \neq 0\} \arrow[rr, bend left, "\varphi_{01}"] & & \{x_0 \neq 0\} \arrow[r, phantom, "\subset"] & \mathbb{A}^2 \arrow[d, equal] \\
\{(x_1, x_2)\} & & & & \{(x_0, x_2)\}
\end{tikzcd}$$

$$(x_1, x_2) \xrightarrow{\varphi_0^{-1}} [1 : x_1 : x_2] \xrightarrow{\varphi_1} \left(\frac{1}{x_1}, \frac{x_2}{x_1} \right).$$

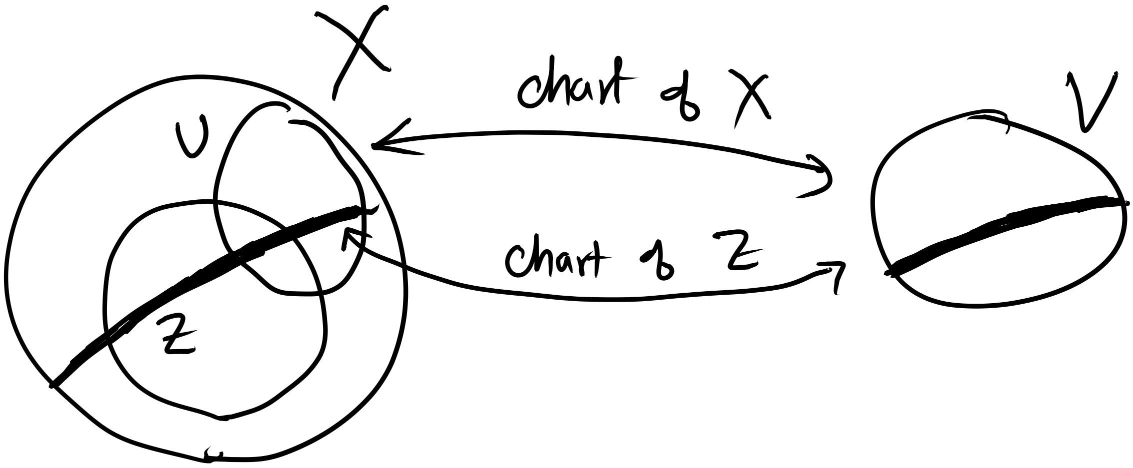
so $\varphi_{01} : \mathbb{A}^2 \setminus V(x_1) \longrightarrow \mathbb{A}^2 \setminus V(x_0)$

$$(x_1, x_2) \longmapsto \left(\frac{1}{x_1}, \frac{x_2}{x_1} \right)$$

is regular.

Open & closed subvarieties

Let X be an algebraic variety, and $Z \subset X$ a closed subset. Then Z is naturally an algebraic variety. The atlas for Z is induced from the atlas for X. i.e. if $\varphi: U \to V$ is a chart for X, then $\varphi_Z : U \cap Z \to \varphi(U \cap Z)$ is a chart for Z. Note that the $\varphi(U \cap Z)$ is a closed subset of the affine variety V, so it is itself an affine variety.



Similarly, if $Y \subset X$ is an open subset, then Y is naturally an algebraic variety. The charts are again obtained by restricting the charts of X.

$\varphi : U \to V$ chart of X

$\varphi_Y : U \cap Y \to \varphi(U \cap Y)$ chart of Y.

(Note: $\varphi(U \cap Y)$ is a quasi-affine, so the latter is not necessarily an affine chart, only a "quasi-affine chart". But we can always write a quasi affine as a union of open affines, & these will provide affine charts).

<u>Def</u>: A <u>Projective Variety</u> is a closed subset of $\mathbb{P}^n$.

A <u>quasi-projective variety</u> is an open subset of a projective variety

Examples:

① $X = V(X_0X_1 - X_2^2) \subset \mathbb{P}^2$

Three affine charts, corresponding to $X_0 \neq 0$, $X_1 \neq 0$, and $X_2 \neq 0$.

Chart $(X_0 \neq 0) = U_0 \cong \mathbb{A}^2$

$$X \cap U_0 \subset U_0 \cong \mathbb{A}^2$$
$$\| $$
$$V(x_1 - x_2^2).$$

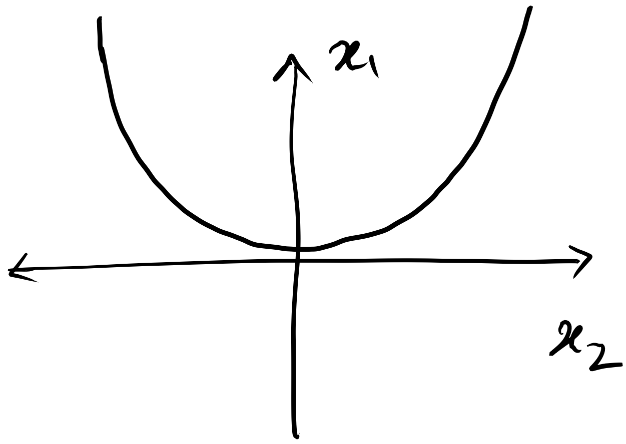
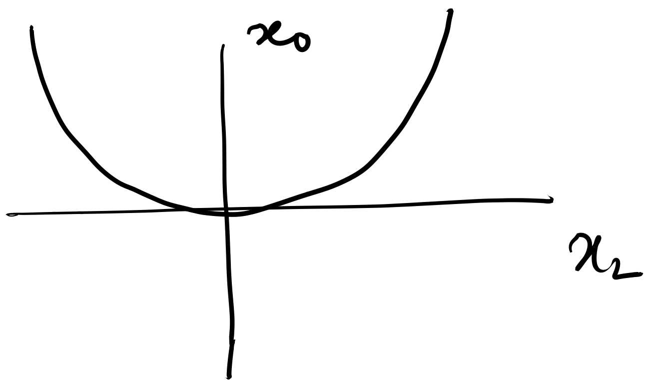


Chart: $: (X_1 \neq 0) = U_1 \cong \mathbb{A}^2$

$$X \cap U_1 \subset U_1 \cong \mathbb{A}^2$$
$$\|$$
$$V(x_0 - x_2^2)$$

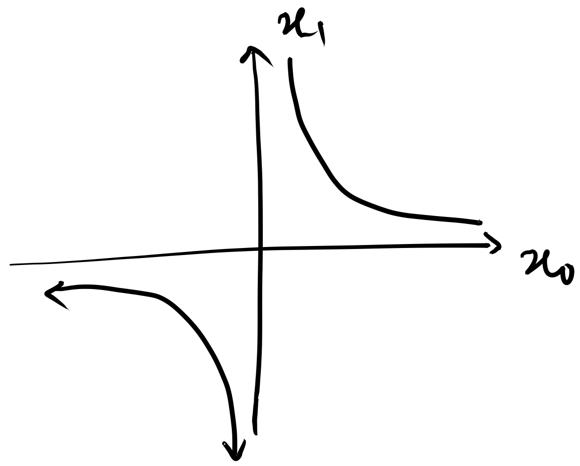


<u>Chart</u> $(X_2 \neq 0) = U_2 \cong \mathbb{A}^2$

$X \cap U_2 \subset U_2$

$\quad \parallel$

$V(x_0 x_1 - 1)$



Missing from Chart 1 :-

$$\mathbb{P}^2 \setminus U_0 = \{ [0 : X_1 : X_2] \}$$
$$\cong \quad \mathbb{P}^1$$

$$V(X_0) \cap X = \{ [0:1:0] \}.$$

So one point of X is missing from Chart 1.

This point is visible in Chart 2, but Chart 2 is missing $[1:0:0]$.

Chart 3 is missing both $[1:0:0]$ & $[0:1:0]$.

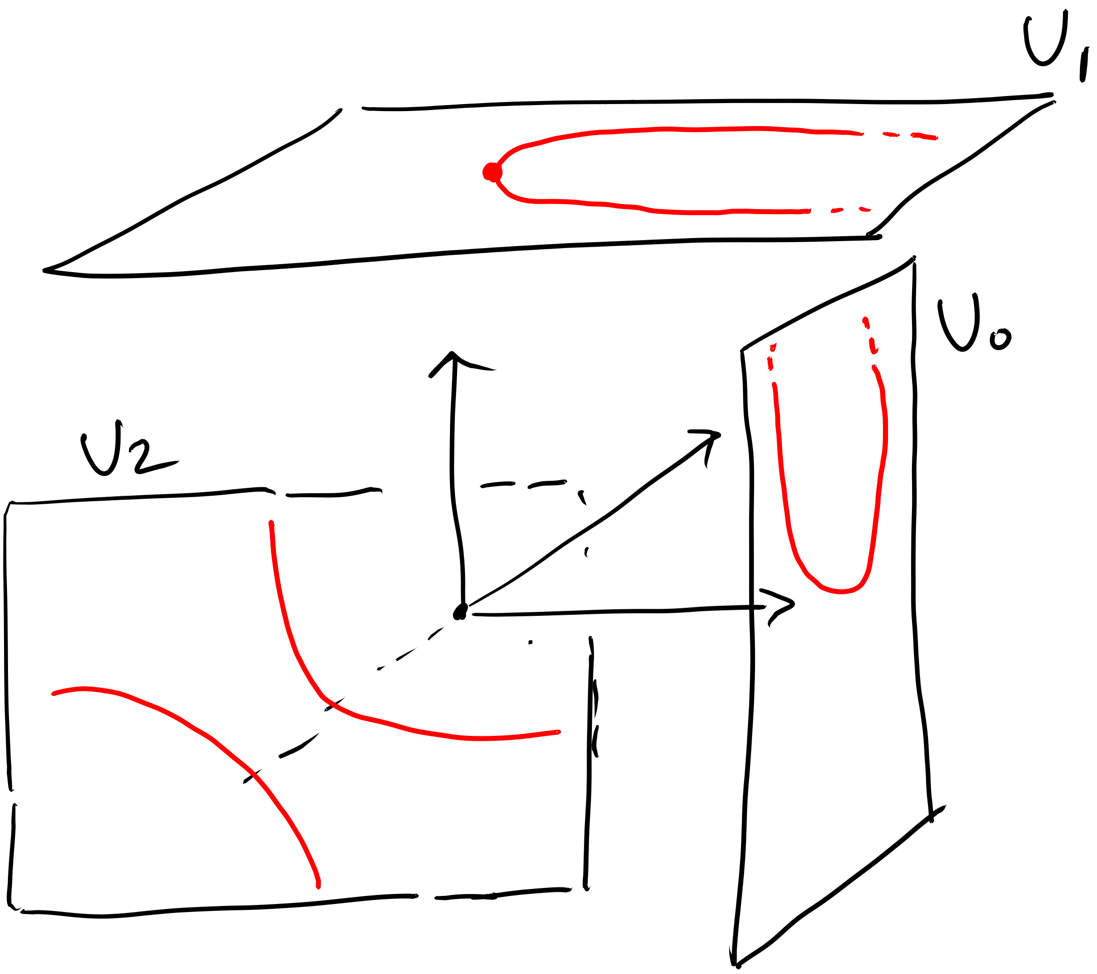
Visualization.

$\mathbb{P}^2$ = Lines in 3-space

$U_0 = \{$Lines meeting $X_0 = 1$ plane$\}$

$U_1 = \{$ Lines meeting $X_1 = 1$ plane$\}$

$U_2 = \{$ lines meeting $X_2 = 1$ plane$\}$.



- Every projective variety is quasi proj.
- Every affine variety is quasi proj.

How?

We have $\mathbb{A}^n \subset \mathbb{P}^n$ open as U_n.

If $X \subset \mathbb{A}^n$ is closed, then

$$X = \overline{X} \cap \mathbb{A}^n$$

$\hookrightarrow$ closure of X in $\mathbb{P}^n$

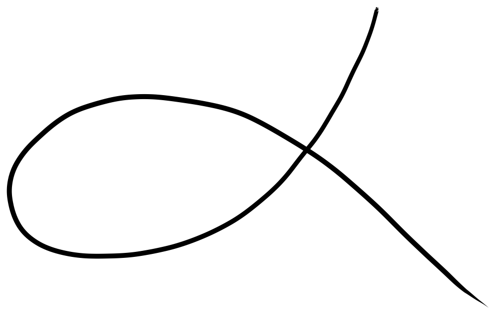
so $X \overset{\text{open}}{\subset} \overline{X} = $ Projective.

So every affine variety is an open in a projective variety.

$\overline{X} \subset \mathbb{P}^n$ is called the <u>projective closure</u> of $X \subset \mathbb{A}^n$.

<u>Example</u>:

$$X = V(y^2 - x^3 - x^2) \subset \mathbb{A}^2$$



nodal cubic

View $\mathbb{A}^2 = \{(x,y)\}$ as

$$\{[x:y:1]\} = \{[X:Y:Z] \mid Z \neq 0\} \subset \mathbb{P}^2.$$

Then the projective closure of X is cut out by the homogenization of $f(x,y) = y^2 - x^3 - x^2$ with respect to Z.

i.e. by $F(X,Y,Z) = Z^3 f\left(\frac{X}{Z}, \frac{Y}{Z}\right)$

$$= (Y^2 Z - X^3 - X^2 Z).$$

"Points at infinity" $= \{\text{Pts with } Z=0\}$
$= \{[0:1:0]\}$

<u>Prop</u> (Exercise) Let $I \subset k[x_0, \dots, x_{n-1}]$ be an ideal. Define

$$I^{hom} = \text{ideal generated by } \{p^{hom} \mid p \in I\} \subset k[x_0, \dots, x_n]$$

where $p^{hom}(x_0, \dots, x_n) = x_n^{\deg p}\, p\left(\frac{x_0}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right)$.

Then the proj. closure of $X = V(I) \subset \mathbb{A}^n$ in $\mathbb{P}^n$ is $\overline{X} = V(I^{hom})$.

Affine cones and the projective nullstellensatz

There is a close connection between Zariski closed subsets of $\mathbb{P}^n$ and homogeneous ideals of $k[x_0, \dots, x_n]$.

Let $f \in k[x_0, \dots, x_n]$ be a polynomial. We can write f uniquely as

$$f = f_0 + f_1 + \dots + f_d$$

where f_i is homog. of degree i. The f_i is called the degree i homog. component of f.

Note that the following are equivalent. (easy)

① $I \subset k[x_0, \dots, x_n]$ is generated by homogeneous polynomials.

② I has the property that if $f \in I$ then all the homog. comp. f_i of f lie in I.

Def: A homog. ideal is an ideal satisfying ① / ②.

Let $X \subset \mathbb{P}^n$ be cut out by a homog ideal I

Consider $CX = V(I) \subset \mathbb{A}^{n+1}$

CX is called the <u>affine cone</u> of X.

It is a cone in the following sense.

<u>Def</u>. $C \subset \mathbb{A}^{n+1}$ is a <u>cone</u> if
$\forall$ $x \in C$ and $\lambda \in k$, $\lambda x \in C$.

Next, suppose $C \subset \mathbb{A}^{n+1}$ is a Zariski closed cone. We claim that $I(C)$ is a homog. ideal. Indeed, if $f(x) \in I(C)$ then $f(\lambda x) \in I(C)$ $\forall$ $\lambda \in k^\times$.

But if $f = f_0 + f_1 + \cdots + f_d$

then

$$f(\lambda x) = f_0 + \lambda f_1 + \cdots + \lambda^d f_d \in I(C)$$

By taking different choices of λ, we see that $f_i \in I(C)$ (we are using k is infinite & the van der monde matrix!)

So if C is a cone and $C \neq \{(0, \cdots, 0)\}$ set $X = V(I(C)) \subset \mathbb{P}^n$. Then $C = CX$.

So we have the bijection

$$\left\{\begin{array}{c}\text{Zar. closed}\\ \text{subvar of } \mathbb{P}^n\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Zariski closed}\\ \text{cones in}\\ \mathbb{A}^{n+1}\end{array}\right\}$$

except $\{(0,\dots,0)\}$

By the Nullstellensatz, we have a bijection

$$\left\{\begin{array}{c}\text{Zar closed cones}\\ \text{in } \mathbb{A}^{n+1}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Homog. radical}\\ \text{ideals.}\end{array}\right\}$$

Putting everything together, we get

<u>The Projective Nullstellensatz</u>

$$\left\{\begin{array}{c}\text{Zar. closed}\\ \text{subsets of } \mathbb{P}^n\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Radical homog.}\\ \text{ideals of}\\ k[X_0,\dots,X_n]\end{array}\right\}$$

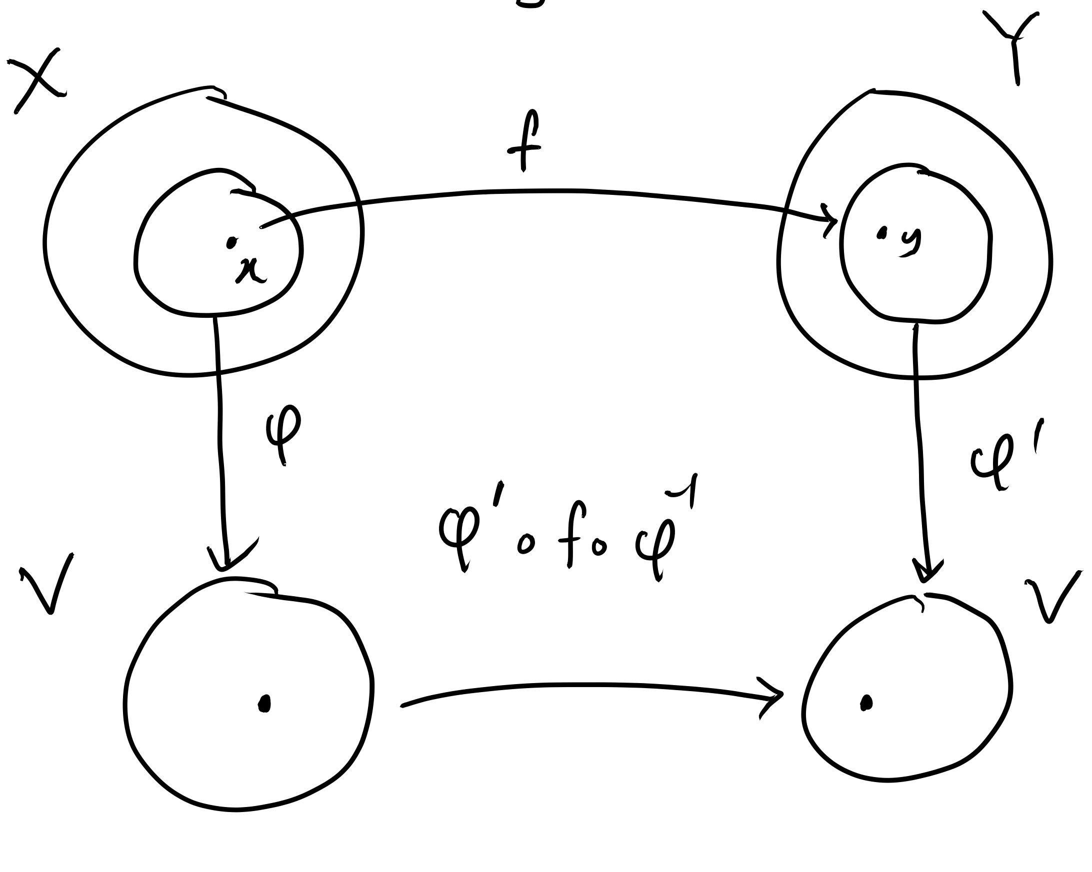
except $(X_0,\dots,X_n)$

"<u>irrelevant ideal</u>"

Regular maps

Let $f : X \to Y$ be a map between algebraic varieties, and let $x \in X$. We say that f is <u>regular</u> at x if it is regular at x in charts of X and Y containing x and $y = f(x)$.

That is, let (U, V, φ) be a chart on X with $x \in U$, and let (U', V', φ') be a chart on Y with $y \in U'$.



We get the map $\varphi' \circ f \circ \varphi^{-1}$ defined on the open subset $\varphi(U \cap f^{-1}(U'))$ of V (this contains $\varphi(x)$).

$$\varphi' \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(U')) \to V'$$

We say that f is <u>regular</u> at x if $\varphi' \circ f \circ \varphi^{-1}$ is regular at $\varphi(x)$.

The definition does not depend on the charts chosen because the transition functions are regular.

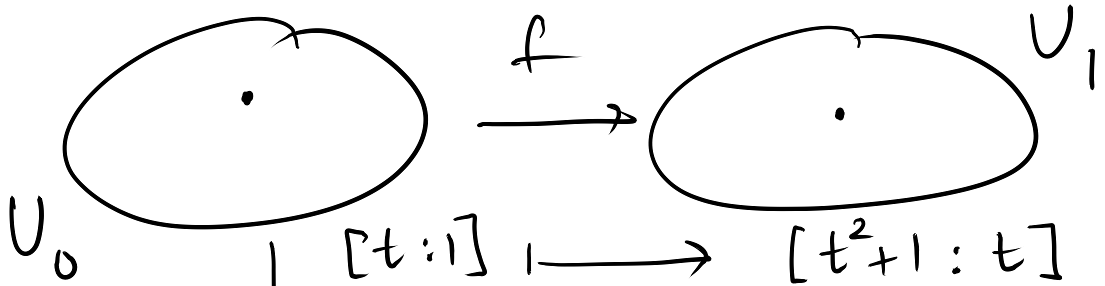
We say f is regular if f is regular at all $x \in X$.

Example.

$$f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$
$$[X:Y] \longmapsto [X^2+Y^2 : XY]$$

Check regular at $[0:1]$

$$[0:1] \overset{f}{\longmapsto} [1:0]$$



$$\begin{tikzcd}
{[t:1]} \arrow[r, mapsto] \arrow[d, shift right] & {[t^2+1:t]} \arrow[d] \\
\mathbb{A}^1 \arrow[r] \arrow[u, shift right] & \mathbb{A}^1
\end{tikzcd}$$

$$t \longmapsto \frac{t}{t^2+1}$$

regular around 0.

(In fact regular everywhere.)

Ex. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(k)$

Then
$$[X:Y] \longmapsto [aX+bY : cX+dY]$$
is an isomorphism $\mathbb{P}^1 \to \mathbb{P}^1$!
The inverse is given by the inverse matrix.

Such a transformation is called a projective linear transformation

Ex. Likewise $M \in GL_{n+1}(k)$ gives an invertible projective linear transformation

$$M : \mathbb{P}^n \to \mathbb{P}^n.$$

$X \subset \mathbb{P}^n$ & $Y \subset \mathbb{P}^n$ are called <u>projectively equivalent</u> if $\exists M$ such that $MX = Y$.

<u>Ex</u>. Any 3 points of $\mathbb{P}^1$ are projectively equivalent to any other 3 points.

More: Given $p, q, r \in \mathbb{P}^1$ $\exists!$ projective linear transform such that

$$0 \mapsto p$$
$$1 \mapsto q$$
$$\infty \mapsto r.$$