Products and the Segre embedding

Let X and Y be algebraic varieties. The product set $X \times Y$ is naturally an algebraic variety in the following way.

Let $\{(U_i, V_i, \varphi_i)\}$ be an atlas for X and $\{(U'_j, V'_j, \varphi'_j)\}$ be an atlas for Y.

The topology on $X \times Y$ is the following. First on $U_i \times U_j'$ we put the topology such that the bijection

$$\varphi_i \times \varphi_j' : U_i \times U_j' \longrightarrow V_i \times V_j'$$

is a homeomorphism. Here, $V_i \times V_j'$ is an affine variety, and has its Zariski topology (which is NOT the product topology). Now there is a UNIQUE way to define the topology on $X \times Y$ so that $U_i \times U_j'$ form an open cover — Declare $Z \subset X \times Y$ to be closed iff $Z \cap (U_i \times U_j')$ is closed for all i, j.

The charts for $X \times Y$ are

$$\varphi_i \times \varphi_j' : U_i \times U_j' \longrightarrow V_i \times V_j'.$$

With this definition, note that the two projections $P_1 : X \times Y \to X$ & $P_2 : X \times Y \to Y$ are regular. Moreover, the product satisfies the correct universal property :-

<u>Proposition</u> : A map $Z \xrightarrow{\varphi} X \times Y$ is regular if and only if the two maps $P_1 \circ \varphi : Z \to X$ and $P_2 \circ \varphi : Z \to Y$ are regular.

<u>Proof</u> : (only if) follows because composites of regular maps are regular.

(if) Suppose $\varphi_1 : Z \to X$ & $\varphi_2 : Z \to Y$ are regular.

Choose a chart $U_i \times U_j'$ of $X \times Y$.

Its preimage is $\varphi_1^{-1}(U_i) \cap \varphi_2^{-1}(U_j') = W$ which is an open in Z. Take an affine chart U in Z & consider the map

$\varphi|_{U \cap W} : U \cap W \to U_i \times U_j'$.

The lhs & rhs are quasi affine (via the chart maps). Suppose $U_i \subset \mathbb{A}^m$ & $U_j' \subset \mathbb{A}^n$. Then $\varphi|_{U \cap W}$ is regular iff its $(m+n)$ coordinate components are regular. But if φ_1 & φ_2 are regular then the first m

& the last n coord. are regular. But then all coordinates are regular.

$\square$

<u>Example</u> (Most important product)

$$\mathbb{P}^n \times \mathbb{P}^m$$
$$= \{([X_0 : \cdots : X_n], [Y_0 : \cdots : Y_m])\}.$$

What are the closed sets?
Def says - $Z \subset \mathbb{P}^n \times \mathbb{P}^m$ is closed iff $Z \cap (\mathbb{A}^n_i \times \mathbb{A}^m_j)$ is closed for the various charts. But there is a more direct description -

<u>Prop</u>:
Closed sets of $\mathbb{P}^n \times \mathbb{P}^m$ are zero sets of <u>bihomogeneous</u> polynomials in $k[x_0, \ldots, x_n, y_0, \ldots, y_m]$.

A polynomial $p(X_0, \ldots, X_m, y_0, \ldots, y_n)$ is bihomogeneous of bidegree (d, e) if

$$p(\lambda x_0, \ldots, \lambda x_n, \mu y_0, \ldots, \mu y_n) = \lambda^d \mu^e \, p(x_0, \ldots, x_n, y_0, \ldots, y_n).$$

e.g. $x_0 y_1^2 + x_1 y_1 y_2$ is bihomog. of bidegree $(1,2)$ in x_0, x_1; y_0, y_1.

Pf of prop (sketch): Very similar to a similar statement about $\mathbb{P}^n$. Let us show that the topology on $\mathbb{P}^n \times \mathbb{P}^m$ defined by taking zero sets of bihom. systems as closed sets restricts to the Zariski topology on the charts.

Let $Z \subset \mathbb{P}^n \times \mathbb{P}^m$ be the zero set of system of bihomog. equations. Then the set $Z \cap \{x_i \neq 0\} \times \{y_j \neq 0\} = Z \cap \mathbb{A}^n \times \mathbb{A}^m$ is cut out in $\mathbb{A}^n \times \mathbb{A}^m$ by the system obtained by dehomogenising (i.e. set $x_i = y_j = 1$), so it is closed.

Conversely if $Z \subset \{x_i \neq 0\} \times \{y_j \neq 0\}$ is a Zariski closed set, then it is the intersection of a set of $\mathbb{P}^n \times \mathbb{P}^m$ with $\{x_i \neq 0\} \times \{y_j \neq 0\}$. This bigger set is the zero set of the system obtained by homogenising wrt x_i & y_j

$$P^{hom}(X_0, \dots, X_m ; Y_0, \dots, Y_n) =$$

$$X_i^{x\text{-deg } P} \; Y_j^{Y\text{-deg } P} \; p\left(\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}, \frac{Y_0}{Y_j}, \dots, \frac{Y_m}{Y_j}\right)$$

$\square$.

<u>Rem</u>: Another description — the topology on $\mathbb{P}^n \times \mathbb{P}^m$ is the quotient topology from the Zariski topology on

$$(\mathbb{A}^{n+1} \setminus \{0\}) \times (\mathbb{A}^{m+1} \setminus \{0\})$$

The Segre embedding.

Example: $s: \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$

$$([x:y],[u:v]) \longmapsto [xu:yu:xv:yv]$$

This map is regular (check on charts).

Image $\subset V(AC-BD)$.

Claim: $s: \mathbb{P}^1 \times \mathbb{P}^1 \to X$ is an isomorphism.

Pf: Inverse

$$[A:B:C:D] \mapsto ([A:B],[A:C])$$

or $([C:D],[A:C])$

or $([A:B],[B:D])$

or $([C:D],[A:C])$

— at least one formula makes sense!

$\square$.

The geometry of a quadric in $\mathbb{P}^3$

(char $k \neq 0$).

We saw in the tutorial that all non-deg. quadric hypersurfaces in $\mathbb{P}^3$ form one iso. class. We just saw that the quadric $V(AD - BC)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. So

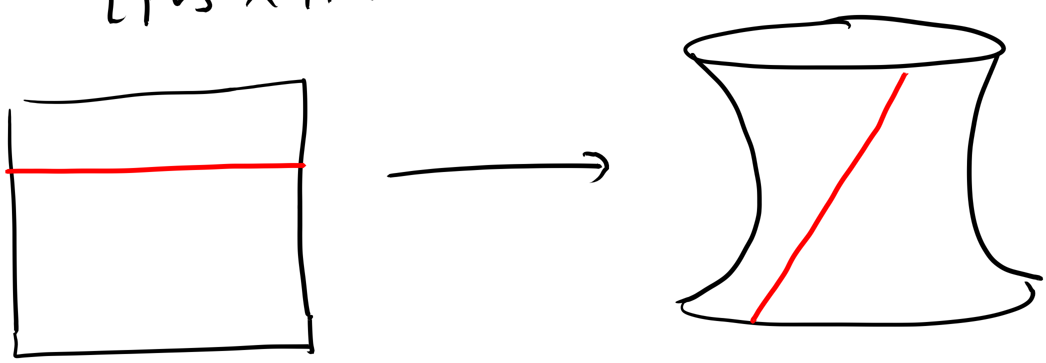
Thm: Any non-deg. quadric hypersurface in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Lines on a quadric:

Let us restrict the map

$$\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow X \subset \mathbb{P}^3$$

to $\{pt\} \times \mathbb{P}^1$.



$$[x_0 : y_0] \times [U : V] \longmapsto [x_0 U : y_0 U : x_0 V : y_0 V]$$

Traces a line as $[U:V]$ varies.

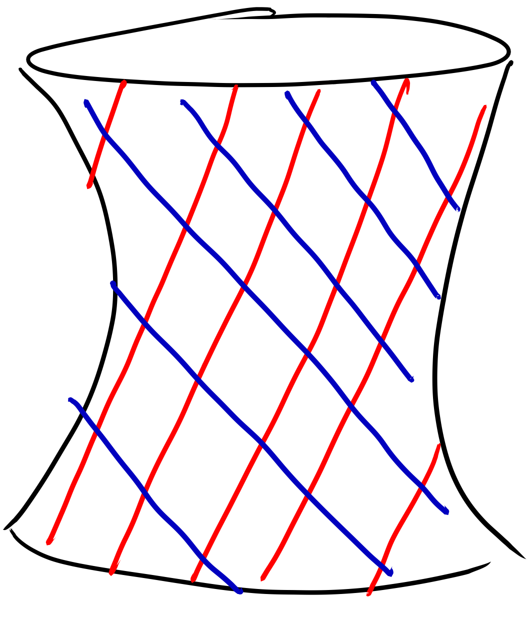
Similarly, if we restrict to $\mathbb{P}^1 \times \{pt\}$, we get

$$[x:y] \times [u_0:v_0] \mapsto [xu_0 : yu_0 : xv_0 : yv_0]$$

also a line.

meets the previous line in a unique point (as expected), namely $[x_0u_0 : y_0u_0 : x_0v_0 : y_0v_0]$.

So $X \subset V(AD-BC)$ is <u>ruled</u> by two families of lines



Segre maps

Consider the map

$$S : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^{(n+1)(m+1)-1}$$

given by $([x_i], [y_j]) \mapsto [x_i y_j])$.

Thm: S is regular. The image of S is the closed subvariety Q of $\mathbb{P}^{(n+1)(m+1)-1} = \{[U_{ij}]\}$ defined by

$$\{U_{ij} \cdot U_{kl} = U_{ik} \cdot U_{jl}\}. \quad — \circledast$$

The map $s: \mathbb{P}^n \times \mathbb{P}^m \to Q$ is an iso.

Cor: $\mathbb{P}^n \times \mathbb{P}^m$ is projective!

Cor: Product of projective varieties is projective.

Cor: Product of quasi-proj varieties is quasi-proj.

Pf of thm: Because s is defined by polynomials, it is regular.

For the rest, it helps to think of a point $[U_{ij}]$ of $\mathbb{P}^{(n+1)(m+1)-1}$ as a matrix

$$U = \begin{bmatrix} U_{00} & \cdots & U_{0n} \\ \vdots & & \\ U_{m0} & \cdots & U_{mn} \end{bmatrix}$$

Then the equations ⊛ are 2×2 minors of U. So Q = Locus of rank 1 matrices

The map s is

$$s: ([X],[Y]) \longmapsto [XY^T]$$

Now, for any rank 1 matrix U, there exist $X \in k^{n+1}\setminus 0$ & $Y \in K^{m+1}\setminus 0$ unique upto scaling such that $U = XY^T$. So s is a bijection. Let $V_{ij} \subset Q$ be the open set of U where $U_{ij} \neq 0$. Then the inverse of s on V_{ij} is given by

$$U \longmapsto (\underbrace{[U_{i0} : \cdots : U_{in}]}_{i^{th}\text{ row}}, \underbrace{[U_{0j} : \cdots : U_{mj}]}_{j^{th}\text{ column}})$$

So the inverse is regular $\square$

The diagonal condition

Consider $\Delta: \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$
$x \mapsto (x, x)$

The image of Δ is closed.

Indeed, image of Δ
$$= \{([x_i], [y_j]) \mid x_i y_j - x_j y_i = 0 \ \forall\, i,j\}.$$

The same is true for any quasi-proj X.
The image of $\Delta: X \to X \times X$ is closed.

The property of the diagonal being closed is important, and plays the role of the Hausdorff condition. It is called separatedness. All quasi-proj varieties are separated.

(HW: For a topological space X, show that X is Hausdroff iff $\Delta: X \to X \times X$ has closed image.)

It leads to the nice conseq. that we should expect from Hausdorff-ness.

For example, we have the following -

Prop (Unique extension property)
Let $U \subset S$ be dense & let X be separated. If $f_1 : S \to X$ & $f_2 : S \to X$ are two regular maps that agree on U, then $f_1 = f_2$.

Pf: Consider $A = \{x \in X \mid f_1(x) = f_2(x)\}$.
We have $U \subset A \subset S$. We want $A = S$. Consider the regular map

$$S \xrightarrow{f = (f_1, f_2)} X \times X$$

Then $A = f^{-1}(\Delta(X))$.
But X separated $\Rightarrow$ $\Delta(X)$ is closed
$\Rightarrow$ A is closed.
U dense $\Rightarrow$ $A = S$.

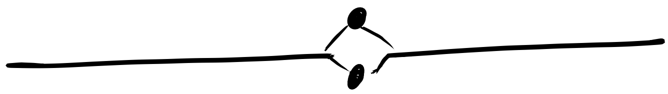
$\square$

Not all algebraic varieties are separated. But many view non-separatedness as a pathology to be avoided.

Example of a <u>non-separated</u> <u>variety</u>

A line with a <u>doubled</u> <u>origin</u>.

Take two copies of $\mathbb{A}^1$ & glue them via the identity on $\mathbb{A}^1 \setminus 0$, but do not glue the two 0's; resulting in something like

$X =$ 

Prop X is non-separated.

Pf: We have two regular maps

$\mathbb{A}^1 \to X \quad (0 \mapsto \text{origin } 1)$

$\mathbb{A}^1 \to X \quad (0 \mapsto \text{origin } 2)$

that agree on $\mathbb{A}^1 \setminus \{0\}$ but are not equal on $\mathbb{A}^1$ $\square$.