The Closed Image Property of Projective Varieties.

Separatedness plays the role of Hausdorffness for algebraic varieties.
What is the analog of compactness?

Def: (Universally closed)
A variety X is universally closed if for every Y, the projection

$$\pi : X \times Y \to Y$$

is a closed map. That is, π must map closed sets to closed sets.

Non-ex: $\mathbb{A}^1$ is not universally closed.

$$\begin{tikzcd} V(xy-1) \arrow[d, "\text{image}"] & \subset & \mathbb{A}^1 \times \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \setminus \{0\} & \subset & \mathbb{A}^1 \end{tikzcd}$$

Not closed.

Prop: X universally closed, Y separated. $f: X \to Y$ a map.

Then $f(X) \subset Y$ is closed.

Pf: Consider the graph

$$\Gamma_f = \{(x, f(x)) \mid x \in X\} \subset X \times Y.$$

Γ_f = Preimage of $\Delta \subset Y \times Y$ under the map $(f, id): X \times Y \to Y \times Y$.

Since Y is separated, $\Delta \subset Y \times Y$ is closed, so $\Gamma_f \subset X \times Y$ is closed.

But $f(X) = \pi(\Gamma_f)$, where $\pi: X \times Y \to Y$ is the projection.

So $f(X) \subset Y$ is closed.

Rem: Easy exercise $f(X)$ is also universally closed.

<u>Thm</u>: Projective varieties are universally closed.

<u>Cor</u>: ① X proj, Y separated
$f : X \to Y \Rightarrow f(X) \subset Y$ closed.

② X proj. $Y = \mathbb{A}^1$
$f : X \to \mathbb{A}^1 \Rightarrow f(X) \subset \mathbb{A}^1$ <u>finite</u>

<u>Pf</u>: Consider $\tilde{f} : X \to \mathbb{P}^1$ obtained by f followed by $\mathbb{A}^1 \subset \mathbb{P}^1$.
Then $f(X) \subset \mathbb{P}^1$ is closed.
So $f(X) = \mathbb{P}^1$ or finite
But $\infty \notin f(X)$ so $f(X) \neq \mathbb{P}^1$
so $f(X)$ must be finite.

③ X proj + connected
$\Rightarrow$ All reg. fun. on X are <u>constant</u>!

(4) X proj, + conn. Y affine
$\Rightarrow$ all $X \to Y$ are constant.

"A proj variety can't map nontrivially to an affine variety."

Applications :-

Consider $V_n = k[X,Y]_n$
$$= \left\{ \sum_{i+j=n} a_{ij} X^i Y^j \right\}$$
$$= (n+1)\text{-dim V-space}$$

$\mathbb{P}V_n \cong \mathbb{P}^n$
$= \{$ Non-zero hom. poly of deg n up to scaling $\}$
$\bigcup$
$\mathcal{D} = \{$ Polynomials with a repeated root $\}$.

Claim: $\mathcal{D} \subset \mathbb{P}^n$ is closed.

Pf: Consider the map

$$m: \mathbb{P}V_1 \times \mathbb{P}V_{n-2} \to \mathbb{P}V_n$$
$$F \times G \mapsto F^2 G.$$

Then m is regular and

$$\mathcal{D} = \text{Image}(m).$$

$\square$.

Translation -

Consider $\sum a_{ij} X^i Y^j = p$

There are some polynomials in a_{ij} whose vanishing is equivalent to p having a double zero.

Ex. $n=2$

$$aX^2 + bXY + cY^2$$

$$\mathcal{D} = V(b^2 - 4ac)$$

Ex. $n=3$

$$aX^3+bX^2Y+cXY^2+dY^3$$

$$D = V(\ldots\ldots\ldots\ldots)$$

extremely complicated!

Same result for triple zeros, quadruple zeros etc.

"Having no repeated roots is a Zariski open condition."

Moving on to more variables.

Say $V_n = k[X,Y,Z]_n$.

$$\mathbb{P}V_n = \mathbb{P}^N \qquad N = \binom{n+2}{2}-1$$

$$\cup$$

$$R = \{\text{Reducible polynomials}\}$$

<u>Claim</u>: $R \subset \mathbb{P}V_n$ is Zariski closed.

<u>Pf</u>: Consider the map

$$m_i : \mathbb{P}V_i \times \mathbb{P}V_{n-i} \to \mathbb{P}V_n$$
$$F \times G \mapsto FG$$

Then $Im(m_i) \subset \mathbb{P}V_n$ is closed,

$$R = Im(m_1) \cup \dots \cup Im(m_{n-1})$$

$Im(m_i)$ = Poly. that factor as $(\deg i) \times (\deg n-i)$.

$\square$

<u>Translation</u> -

Take $p = \sum a_{ijk} x^i y^j z^k$.

There are polynomial equations in a_{ijk} that detect whether p is reducible or irreducible.

"Being irreducible is a <u>Zariski</u> <u>open</u> condition"

Singularities.

Let $F = \sum a_{ijk} x^i y^j z^k$ $\quad C = V(F)$

Def: A point $p \in C$ is called a smooth point or a non-singular point if at least one $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$, or $\frac{\partial F}{\partial z}$ is non-zero at p.

C is called <u>smooth</u> if all its points are smooth.

(motivation - over $\mathbb{C}$
if one of the partials is non-zero, then the implicit function theorem implies that C is a smooth manifold near p).

$$\begin{array}{l} \mathbb{P}V_n \\ \cup \\ U = \{F \mid V(F) \text{ is smooth}\}. \end{array}$$

Then $U \subset \mathbb{P}V_n$ is Zariski open.

"Smoothness is Zariski open."

<u>Pf</u>: Let $Z = \mathbb{P}V_n \setminus U$. We show that Z is closed.

Consider

$$\mathbb{P}^2 \times \mathbb{P}V_n \supset W$$

$$W = \{ (p, F) \mid p \text{ is a sing pt of } V(F) \}.$$

$$= \left\{ ([P_0 : P_1 : P_2], [a_{ijk}]) \;\middle|\; \begin{array}{ll} & \sum a_{ijk} P_0^i P_1^j P_2^k = 0 \\ \frac{\partial}{\partial x} : & \sum i\, a_{ijk} P_0^{i-1} P_1^j P_2^k = 0 \\ \frac{\partial}{\partial y} : & \sum j\, a_{ijk} P_0^i P_1^{j-1} P_2^k = 0 \\ \frac{\partial}{\partial z} : & \sum k\, a_{ijk} P_0^i P_1^j P_2^{k-1} = 0 \end{array} \right\}.$$

$\Rightarrow W$ is closed.

$\Rightarrow Z = \text{Image}(W)$ is closed!

$\square$

Translation - There are polynomial equations in a_{ijk} that detect whether $V(\sum a_{ijk} X^i Y^j Z^k)$ is smooth.

Proof of main theorem: Projective var. are universally closed.

Reductions:—

1. Enough to show $\mathbb{P}^n$ is universally closed.

2. Want: $\pi : \mathbb{P}^n \times Y \to Y$ to send closed sets to closed sets.

Obs: $S \subset Y$ is closed iff for an open cover $\{U_i\}$ of Y, $S \cap U_i \subset U_i$ is closed.

So, by taking an affine cover $\{U_i\}$ of Y, it suffices to show that the maps

$$\mathbb{P}^n \times U_i \to U_i$$

are closed.

③ By ②, suffices to treat the case where Y is affine.

Say $Y \underset{\text{closed}}{\subset} \mathbb{A}^m$.

$$\begin{tikzcd} \mathbb{P}^n \times Y \arrow[r, hook, "\text{closed}"'] \arrow[d] & \mathbb{P}^n \times \mathbb{A}^m \arrow[d] \\ Y \arrow[r, hook, "\text{closed}"'] & \mathbb{A}^m \end{tikzcd}$$

A closed $Z \subset \mathbb{P}^n \times Y$ maps to a closed in Y iff $Z \subset \mathbb{P}^n \times \mathbb{A}^m$, which is also closed, maps to a closed in $\mathbb{A}^m$!

So, it suffices to show that

$$\mathbb{P}^n \times \mathbb{A}^m \longrightarrow \mathbb{A}^m$$

is a closed map.

(4) (Hard work step).

$Z \subset \mathbb{P}^n \times \mathbb{A}^m$ closed.

So $Z = V\big(F_1(x_0, \dots, x_n; t_1, \dots, t_m),$
$F_2(x_0, \dots, x_n; t_1, \dots, t_m),$
$\vdots$
$F_\ell(x_0, \dots, x_n; t_1, \dots, t_m) \big)$

where the F_i's are homogeneous in x's.

Image of Z

$$= \left\{ \underline{t} \;\middle|\; \begin{array}{l} F_1(x,t) = 0 \\ \vdots \\ F_\ell(x,t) = 0 \end{array} \text{ has a solution in } \mathbb{P}^n \right\}.$$

Why is this Zariski closed?

Let's show that the complement is open.

Suppose $a \in \mathbb{A}^m$ is such that $\{F_i(x;a)\}$ has no solution in $\mathbb{P}^n$. Let $I_a = \langle F_i(x;a) \rangle$, $I_a \subset k[x_0, \dots, x_n]$ ideal. By the projective Nullstellensatz, $\sqrt{I_a} = (x_0, \dots, x_m)$ or (1).

In any case $(x_0, \dots, x_m)^N \subset I_a$ for some N.

We claim that $(x_0, \dots, x_m)^N \subset I_t$ for all t in a Zariski neighborhood of a.

To prove the claim, let $d_i = \deg F_i$.
Set $V_d = k[x_0, \dots, x_n]_d$

Consider the map of fin dim. v. spaces

$$M_t : V_{N-d_1} \oplus \cdots \oplus V_{N-d_e} \to V_N$$

$$(G_1, \dots, G_e) \mapsto F_1(x,t) G_1(x) + \cdots + F_e(x,t) G_e(x).$$

This map depends polynomially on t. That is, if you represent it as a matrix, its entries are polynomials in t.

Now for $t=a$, M_t has full rank (it is surjective).

$\Rightarrow \exists$ non-zero minor of size $\dim V_H \times \dim V_N$ in M_t. For $t=a$. Consider the Zariski open $U \subset \mathbb{A}^m$ defined by the non-vanishing of this minor. Then $a \in U$ and for all $t \in U$, the map M_t is surjective.

So for all $t \in V$, the ideal I_a contains $(x_0, \dots, x_n)^N$, so $t \notin \mathrm{Im}(Z)$.

$\square$.