Irreducibility

Let X be a topological space.
We say X is <u>reducible</u> if

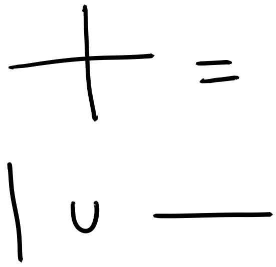
$$X = Y \cup Z$$

where $Y \subset X$ and $Z \subset X$ are proper closed subsets.

If X is not reducible, it is <u>irreducible</u>.

<u>Examples</u>:

$$X = V(xy) \subset \mathbb{A}^2.$$
$$= V(x) \cup V(y)$$



so X is reducible.

What about $X = V(xy-1)$?

<u>Prop</u>: Let $X \subset \mathbb{A}^n$ be Zariski closed.
The following are equivalent.

(1) X is irreducible

(2) $I(X)$ is a prime ideal

(3) $k[X]$ is an integral domain.

Proof: (2) $\Leftrightarrow$ (3) we know from algebra.

Let us show (1) $\Leftrightarrow$ (2).

Equivalently, X reducible $\Leftrightarrow$ $I(X)$ not a prime ideal.

($\Rightarrow$) $X = Y \cup Z$, $Y, Z \subset X$ proper closed.

$Y \subsetneq X \Rightarrow I(Y) \supsetneq I(X)$.

$Z \subsetneq X \Rightarrow I(Z) \supsetneq I(X)$

$Y \cup Z = X \Rightarrow I(Y) \cap I(Z) = I(X)$.

Choose $f \in I(Y) \setminus I(X)$

$g \in I(Z) \setminus I(X)$

Then $fg \in I(Y) \cap I(Z) = I(X)$.

so $I(X)$ is not prime.

($\Leftarrow$) Let $f, g \notin I(X)$ but $fg \in I(X)$.

Set $Y = V(I + f) \subsetneq X$

$Z = V(I + g) \subsetneq X$

But $Y \cup Z = X$

$\square$

X irreducible $\Leftrightarrow$

Any open in X is dense $\Leftrightarrow$

Any two nonempty opens have a nonempty intersection.

<u>Prop</u>: X an irred. top. space & $f: X \to Y$ a continuous map. Then $f(X)$ is irred.

<u>Prop</u>: $U \subset X$ a dense subset. If U is irred, then X is irred.

<u>Cor</u>: $\mathbb{A}^n$, $\mathbb{P}^n$ are irred.

<u>Prop</u>: X, Y irreducible $\Rightarrow X \times Y$ is irreducible.

Which topology? Any where $X \times \{y\} \to X$ and $\{x\} \times Y \to Y$ are homeomorphisms.

Pf: Suppose

$$X \times Y = A \cup B$$

$A, B \subset X \times Y$ closed.

Want $A = X \times Y$ or $B = X \times Y$.

For every $x \in X$, we have

$$\{x\} \times Y = A \cap \{x\} \times Y \cup B \cap \{x\} \times Y$$

Since Y is irreducible $A \cap \{x\} \times Y = \{x\} \times Y$ or $B \cap \{x\} \cap Y = \{x\} \times Y$.

Let $\alpha \subset X$ consist of $x \in X$ s.t.
$A \cap \{x\} \times Y = \{x\} \times Y$ (i.e. $\{x\} \times Y \subset A$)
& similarly $\beta \subset Y$ consist of $x \in X$ s.t.
$B \cap \{x\} \times Y = \{x\} \times Y$ (i.e. $\{x\} \times Y \subset B$).

Claim: $\alpha, \beta \subset X$ are closed.

Pf:
$$\begin{aligned}\alpha &= \{x \in X \mid (x,y) \in A \ \forall y \in Y\} \\ &= \bigcap_{y \in Y} \{x \in X \mid (x,y) \in A\} \\ &= \bigcap_{y \in Y} \underbrace{\pi_X(A \cap X \times \{y\})}_{\text{closed}}\end{aligned}$$

$\square$

Note that $\alpha \cup \beta = X$.

Since X is irreducible, we must have

$\alpha = X$ or $\beta = X$.

So $A = X \times Y$ or $B = X \times Y$

$\square$

Suppose X is reducible

$$\begin{array}{ccccccc}
 & & & X & & & \\
 & & & \| & & & \\
 & & & Y \cup Z & & & \\
 & & // & & \backslash\backslash & & \\
 & & Y_1 \cup Y_2 & & Z_1 \cup Z_2 & & \\
 & // & \backslash\backslash & & // & \backslash\backslash & \\
 \cdots & & \cdots & & \cdots & & \cdots
\end{array}$$

continue

Will this stop?

Def: A topological space X is <u>Noetherian</u> if every descending chain of closed subspaces of X stabilizes.

X affine variety $\Rightarrow$ X Noetherian

Pf: Follows from the fact that $k[X]$ is a Noetherian ring.

$\square$

Easy: Finite unions of Noetherian spaces are Noetherian.

Consq :- Quasi-proj. varieties are Noetherian (because they have a finite affine cover)

Hence, every quasi proj. X can be written as

$X = X_1 \cup \cdots \cup X_n$ where $X_i \subset X$ is closed and irreducible. Further, suppose each X_i is a maximal irreducible closed subset of X.

Claim: There is a unique decomp of X as a union of maximal closed irred subsets.

Pf: $X = X_1 \cup \cdots \cup X_n$
$= X_1' \cup \cdots \cup X_m'$.

We show $X_1' = X_i$ for some i.

$$X_1' = \underbrace{(X_1 \cap X_1')}_{} \cup \cdots \cup \underbrace{(X_n \cap X_1')}_{}$$

closed in X_1' & X_1' irred.

$\Rightarrow$ $X_1' \cap X_i = X_1'$ for some i.

i.e. $X_1' \subset X_i$ for some i.

X_1' maximal $\Rightarrow$ $X_1' = X_i$.

Now remove X_1' & X_i & continue.

$\square$.

Translation into algebra (for affine X)

Let $I \subset k[X_1, \ldots, X_n]$ be a radical ideal.

Then I has a unique expression.

$$I = P_1 \cap \cdots \cap P_n$$

where P_i are prime ideals &

$P_i \not\subset P_j$ for $i \neq j$.

Rational functions and rational maps

In this section, all varieties are separated.

Let X be irreducible.
Consider pairs (U, f) where $U \subset X$ is a non-empty open and $f: U \to \mathbb{A}^1$ is a regular function.
Call (U, f) & (V, g) *equivalent* if
$$f|_{U \cap V} = g|_{U \cap V}.$$

A *rational function* on X is an equivalence class of pairs (U, f) under the equivalence above.

Ex. $X = \mathbb{A}^1$.
$(X, \text{polynomial in } t)$ is a rat. fun.
$(\mathbb{A}^1, t^2+1) \sim (\mathbb{A}^1 \setminus \{0\}, t^2+1)$.

But there are more:
$(\mathbb{A}^1 \setminus \{0\}, \frac{1}{t})$
$(\mathbb{A}^1 \setminus \{0, 1\}, \frac{t}{t^2-1})$.

More generally

$$\left(\mathbb{A}^1 - V(g),\ \frac{f(t)}{g(t)}\right).$$

- Rat. functions on X form a field denoted by $k(X)$.

  $k(\mathbb{A}^1) = k(t)$.

  More generally, if X is affine, then $k(X) \cong \operatorname{frac}(k[X])$

- $U \subset X$ open $\Rightarrow$ $k(U) \cong k(X)$.

- $$\begin{aligned} k(\mathbb{P}^n) &= k(\mathbb{A}^n) \\ &= k\left(\frac{x_0}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right) \\ &= \left\{ \frac{F(x_0, \dots, x_n)}{G(x_0, \dots, x_n)} \;\middle|\; F, G \text{ homog of the same degree} \right\} \\ &=: k(x_0, \dots, x_n)_0 \end{aligned}$$

Unfortunately $X \mapsto k(X)$ is not a functor.

$\varphi : X \to Y$, $\quad f \in k(Y)$

want to define $\varphi^* f \in k(X)$.

Try $\varphi^* f = f \circ \varphi$.

But what if f is defined on an open $U \subset Y$ & $f(x) \subset U^c$?

This is a problem, but it goes away if we restrict to φ that have the following property - $\varphi(X) \subset Y$ is dense.

Such maps are called <u>dominant</u>.

Then $\varphi^* f := f \circ \varphi$ makes sense.

More precisely if f is regular on $U \subset Y$ then $\varphi \circ f$ is regular on $f^{-1}(U) \subset X$

So we set

$$\varphi^* (U, f) = (f^{-1}(U), f \circ \varphi)$$

& we get

$$\varphi^* : k(Y) \to k[X].$$

A <u>rational map</u> from X to Y is an equivalence class of pairs (U, f) where $U \subset X$ is open & $f: U \to Y$ is regular. A rational map represented by (U, f) is often denoted by

$$f: X \dashrightarrow Y.$$

The U is left out of the notation.