All varieties today are irred & quasi-proj

- Rational map $f : X \dashrightarrow Y$ is $(U, f)$ where $U \subset X$ is open & $f : U \to Y$ is regular / eqv.

- $f : X \dashrightarrow Y$ is dominant if $f(U) \subset Y$ is dense.

  This property does not depend on the chosen $U$.

- $f : X \dashrightarrow Y$ dominant induces a map $f^* : k(Y) \to k(X)$

- Conversely a map $k(Y) \to k(X)$ arises from a unique dominant rational map $X \dashrightarrow Y$.

So

Irred var & dom. rat. maps $\overset{\sim}{\longleftrightarrow}$ finitely generated fields / $k$.

We have many more sophisticated tools to analyze projective varieties than we have to analyze fields.

<u>Q</u>: Given a fin. gen. field $/k$ find <u>the</u> <u>nicest</u> variety with that as its fraction field. (called "model".)

<u>A</u>: (In progress) - Minimal model program.

<u>Def</u>: A dominant rat. map $f : X \dashrightarrow Y$ is <u>birational</u> if $\exists$ $g : Y \dashrightarrow X$ (dominant rat) s.t. $f \circ g = id$, $g \circ f = id$.

We say $X$ & $Y$ are birational or birationally isomorphic if a birational $f : X \dashrightarrow Y$ exists.

<u>Obs</u>: $f : X \dashrightarrow Y$ is birat iff $f^* : k(Y) \to k(X)$ is an isomorphism.

So $X$ & $Y$ are birational iff $k(X)$ & $k(Y)$ are isomorphic over $k$.

$$
\begin{tikzcd}
k(X) \arrow[r, shift left] & k(Y) \arrow[l, shift left] \\
k \arrow[u, hook] \arrow[r, equal] & k \arrow[u, hook]
\end{tikzcd}
$$

Examples:

① $\mathbb{P}^n \sim \mathbb{A}^n$ ($\sim$ = bir. iso. Not standard notation)

② $U \subset X$ open
$\Rightarrow U \sim X$.

③
$$
\begin{array}{ccc}
\mathbb{P}^1 \times \mathbb{P}^1 & \sim & \mathbb{P}^2 \\
\cup & & \cup \\
\mathbb{A}^1 \times \mathbb{A}^1 & = & \mathbb{A}^2
\end{array}
$$

Thm: (Char $k = 0$) Every variety is birat. to a hypersurface in affine/proj space.

Pf: Take $X$. Then $k(X)$ can be written as a finite ext. of $k(t_1, \ldots, t_n)$.
By the primitive element theorem

$$k(X) = k(t_1, \ldots, t_n)[x] \big/ p_t(x)$$

where $P_t(x) \in k(t_1, \dots, t_n)[x]$ irred.

Clear denominators so that

$$p_t(x) \in k[t_1, \dots, t_n, x] \text{. irred.}$$

Then $k(X) = \text{frac } k[t_1, \dots, t_n, x] / P_t(x)$

$$= k\Big(\underbrace{V(P_t(x))}_{Y}\Big)$$

so $X \sim$ hypersurface $Y$.

$\square$.

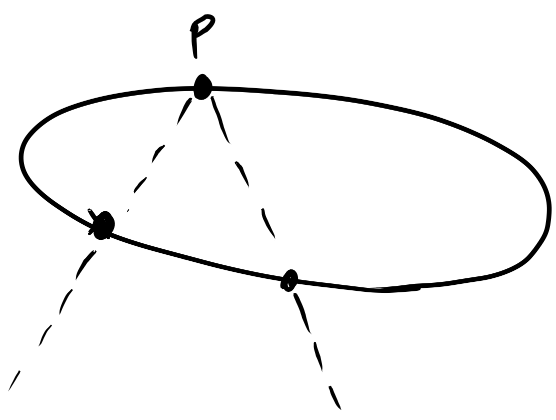
$$X = V(F) \subset \mathbb{P}^n$$

$F$ homog. of degree $d$.

$d = 1 \Rightarrow X \cong \mathbb{P}^{n-1}$.

$d = 2 \Rightarrow X \sim \mathbb{P}^{n-1}$.

Project from $p$ to $\mathbb{P}^{n-1}$.




$X \dashrightarrow \mathbb{P}^{n-1}$

birat. map

In coordinates

$$X = V(x_0x_1 - x_2x_3 + x_4^2 + \cdots + x_k^2)$$

$$\subset \mathbb{P}^n$$

$$p = [1:0:0:\cdots\cdots:0]$$

$$\pi : X \dashrightarrow \mathbb{P}^{n-1}$$

$$[x_0 : \cdots\cdots : x_n] \longmapsto [x_1 : \cdots\cdots : x_n].$$

inverse

$$\tau : [x_1 : \cdots\cdots : x_n] \longmapsto \left[\frac{x_2x_3}{x_1} : x_1 : \cdots : x_n\right]$$

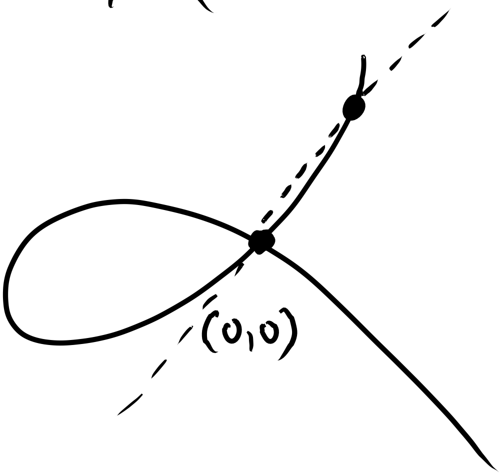
So every (irreducible) quadric hypersurface is $\underbrace{\text{birational to projective space}}_{\text{"rational"}}$.

Cubics:

$C \subset \mathbb{P}^2$ $\qquad$ $C$ "singular"!

$C = V(Y^2Z - X^3 - X^2Y)$

affine $\quad V(y^2 - x^3 - x^2)$




Project: $\pi_{(0,0)} : C \dashrightarrow \mathbb{P}^1$

$[X:Y:Z] \dashrightarrow [X:Y]$

$(x,y) \dashrightarrow \frac{x}{y}$

$(t^2-1, t(t^2-1)) \dashleftarrow t$

Calculation

$y = xt$

$x^2t^2 - x^3 - x^2 = 0 \qquad x^2(t^2 - x - 1) = 0$

$x =$

**Thm:** A smooth plane cubic $C \subset \mathbb{P}^2$ is **NOT** birational to $\mathbb{P}^1$.

**Cubic Surfaces.**

$$S = V(\text{cubic}) \subset \mathbb{P}^3.$$

**Thm:** A smooth cubic surface **is** birational to $\mathbb{P}^2$

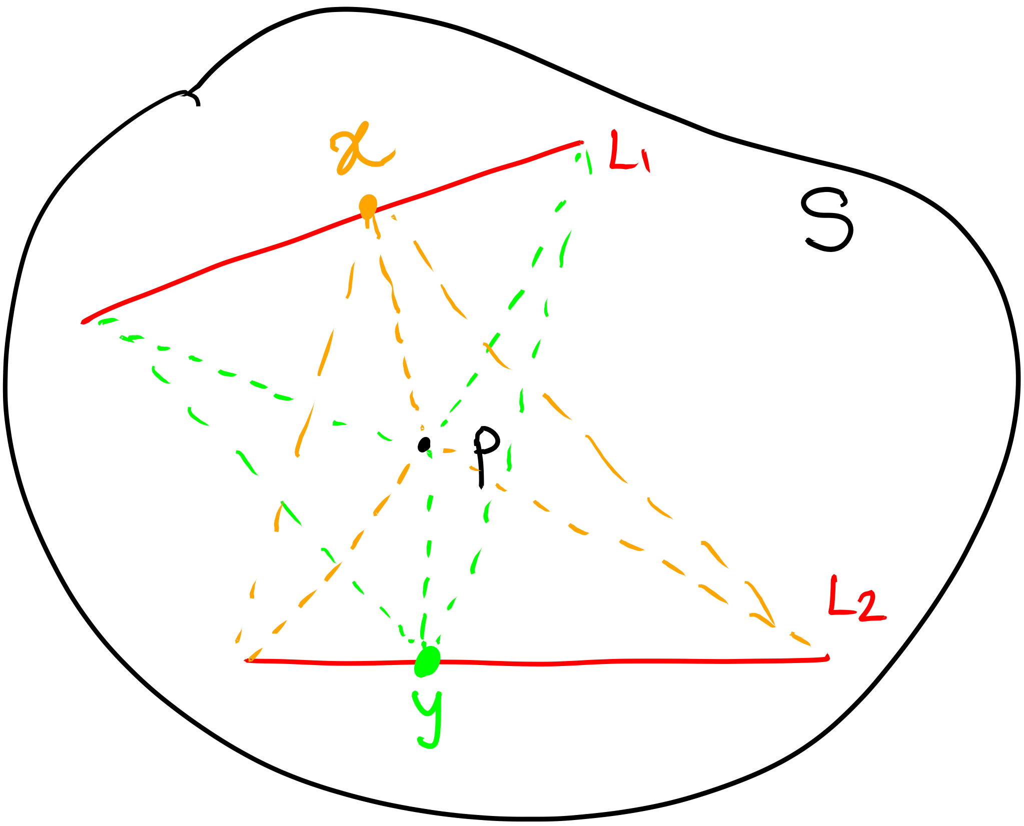
**Pf:** (Sketch – you can fill in the details for a cubic surface you know, e.g. Fermat cubic).

① $S$ contains two skew lines $L_1$ & $L_2$.

② $\pi : S \dashrightarrow L_1 \times L_2$

$$p \mapsto (\mathrm{Span}(L_1, p) \cap L_2, \ \mathrm{span}(L_2, p) \cap L_1)$$

Third pt of intersection of $\mathrm{Span}(x, y) \cap S$. $\longleftarrow\!\!\shortmid \ (x, y)$




$x - p - y$ are collinear.

Q: $T = V(\text{cubic}) \subset \mathbb{P}^4$

Thm (Clemens-Griffiths) A smooth cubic 3-fold is NOT rational.

Q: $F = V(\text{cubic}) \subset \mathbb{P}^5$ & higher

Rational or not? OPEN Q.