Dimension:

motivation for the definition.

X irred. variety.

$$k \subset \underbrace{k(t_1, \dots, t_n) \subset k(X)}_{\text{algebraic.}}$$

$X \dashrightarrow \mathbb{A}^n$ dominant.

For simplicity assume $X \to \mathbb{A}^n$, and also assume X affine, say $X \subset \mathbb{A}^m$

Then $k[X] = k[x_1, \dots, x_m]/I$.

$$\begin{aligned} X &\to \mathbb{A}^n \\ x &\mapsto (f_1(x), \dots, f_n(x)) \end{aligned}$$

Then $k(t_1, \dots, t_n) \to k(X)$ is given by

$$t_i \mapsto f_i(x).$$

Now, if this extension is algebraic, then each x_i satisfies a polynomial equation

$$P_{n,i}(t)\, x_i^n + \cdots + P_{i,0}(t) = 0$$

So, at least over the open subset of $\mathbb{A}^n$ where these polynomials are non-trivial, there are only finitely many possible values of $x_1, \ldots, x_m$ for a given $(t_1, \ldots, t_n)$.

That is, the map

$$\pi : X \to \mathbb{A}^n$$

is "generically finite". ($\exists$ open $U \subset \mathbb{A}^n$ s.t.

$$\pi^{-1}(U) \to U$$

has finite fibers.)

So it makes sense to set

$$\dim X = n.$$

Prop: $\dim(X \times Y) = \dim X \times \dim Y$.

Pf: Shaf 1.33

Prop: $X \subseteq Y$ closed $\Rightarrow \dim X \leq \dim Y$

If equality holds, then $X = Y$.

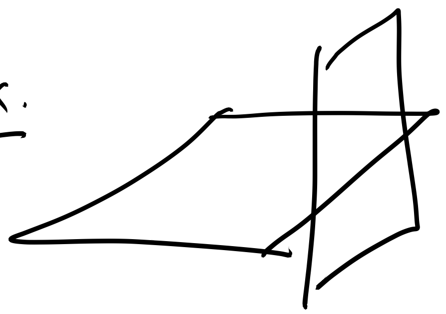
Pf: Shaf Thm 1.19

Hypersurfaces:

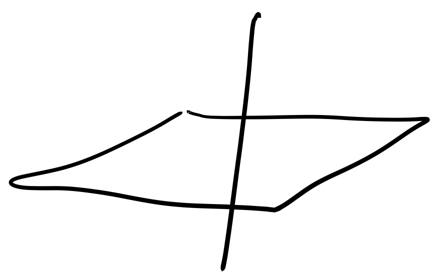
Def: X has dim n if $n =$ max dim of irred. comp. of X.

X has pure dim n if all irr comp of X have dim n.

Ex.



pure dim 2



dim 2

but not pure.

<u>Thm</u>: All hypersurfaces in $\mathbb{A}^n$ or $\mathbb{P}^n$ have pure dim $(n-1)$.

<u>Pf</u>: Shaf Thm 1.20

<u>Thm</u>: Conversely if $X \subset \mathbb{A}^n$ has pure dim $(n-1)$ then $I(X)$ is principal. In particular, X is a hypsurf.

<u>Pf</u>: Shaf Thm 1.21

The same statement holds for $\mathbb{P}^n$. If $X \subset \mathbb{P}^n$ has pure dim $(n-1)$, then $X = V(F)$ for some homog. F. The proof is either by passing to an affine chart or passing to the cone & applying the theorem for $\mathbb{A}^n$ (or $\mathbb{A}^{n+1}$).

What can we say about the subvariety of a general affine variety X obtained by imposing 1 equation?

<u>Thm</u> (Principal Ideal Thm, Hauptidealsatz).
Let X be an irreducible affine of dim n and $f \in k[X]$ non-zero. Then $V(f)$ is either ϕ or has pure dim $(n-1)$.

<u>Pf</u> - We will NOT prove this. But a complete proof is in Shafarevich. It takes some work.

<u>Rem</u>: $V(f) = \phi$ is possible.

<u>Ex</u>. $X = V(xy-1) \subset \mathbb{A}^2$

$f \in k[X]$ is $f(x,y) = x$.

For (quasi) projective varieties, the thm is analogous -

Thm $X \subset \mathbb{P}^N$ quasi proj irred. of dim n. F homogeneous poly in $x_0, \dots, x_N$ not identically 0 on X. Then $V(F) \cap X$ is either ϕ or of pure dim $(n-1)$.

Furthermore if X is projective and $n > 0$, then $V(F) \cap X$ is non-empty.

Pf: Except for the non-emptyness assertion, the rest follows from the affine case by passing to charts.

Let X be proj. & $\dim X > 0$. Let's show $V(F) \cap X$ is non-empty.

Suppose $V(F) \cap X = \phi$. Let's show $\dim X = 0$.

Let $d = \deg F$. Then $\frac{x_i^d}{F}$ is a reg. fun on X. But X is proj (& connected). So this must be a constant. Therefore, all $\frac{x_i^d}{x_j^d}$ are constant on $X \Rightarrow \frac{x_i}{x_j}$ are const.

But $k(X)$ is generated by $\frac{x_i}{x_j}$, so

$k(X) = k \Rightarrow \dim X = 0$

$\square$.

The principal ideal thm has several useful conseq.

① Let $F_1, \dots, F_m$ be homog. poly on $\mathbb{P}^n$ with $m \leq n$.

Then every comp. of $V(F_1, \dots, F_m)$ has $\dim \geq n-m$ & $V(F_1, \dots, F_m)$ is non $\emptyset$.

Pf: Induct on m.

② There does not exist a regular map $\mathbb{P}^n \to \mathbb{P}^m$ for $n > m$.

Pf: Consider a rat map $\mathbb{P}^n \overset{f}{\dashrightarrow} \mathbb{P}^m$ for any n, m. Locally, on some open, it is given by

$$x \mapsto [F_0(x) : \dots : F_m(x)] \quad (*)$$

for some homog. poly F_i.

Suppose $\gcd(F_i) = 1$ (otherwise, take out the common factor.)

We claim that ⊛ defines f globally.

More precisely, $y \in \mathbb{P}^n$ lies in the domain of definition of f iff $F_i(y) \neq 0$ for some i, & $f = [F_0 : \cdots : F_m]$ in a neighborhood of y.

To see this, suppose $y \in \mathrm{dom}(f)$.

Then $f = [G_0 : \cdots : G_m]$ around y for some homog G_i, not all 0 at y.

But then $F_i G_j - G_i F_j = 0$ as a polynomial. We claim that unique factorization implies F_i divides G_i. Indeed, let p be an irreducible factor of F_i. Then $\exists j$ s.t. p does not divide F_j. Since $F_i G_j = F_j G_i$, the power of p dividing F_i also divides G_i.

So F_i divides G_i. Writing $F_i H_i = G_i$, the equation $F_i G_j = F_j G_i \Rightarrow H_i = H_j$, so

$$[G_0 : \cdots : G_m] = H[F_0 : \cdots : F_m] \quad \text{for some } H.$$

So if $\exists$ G_j non vanishing at x, then F_j must also not vanish at x & $f = [F_0 : \cdots : F_m]$ around x.

Now if $m < n$, then the F_i have a common zero & hence f cannot be regular.

$\square$