Grassmannians

Let V be an n-dim vector space.

We saw that

$$Gr(r, V) = \{ r\text{-dim subspaces of } V \}$$

is an algebraic variety,
irreducible of dim $r(n-r)$
projective.

Some remarks:

- $Gr(1, V) = \mathbb{P}V$
- We have a duality isomorphism
$$\begin{aligned} Gr(r, V) &= Gr(n-r, V^*) \\ \lambda &\longmapsto \{ f \in V^* \mid f(\lambda) = 0 \}. \end{aligned}$$
- Suppose $W \subset V$ is a subspace of dim m.
Then $\{ \lambda \in Gr(r, V) \mid \lambda \subset W \} \subset Gr(r, V)$
is isomorphic to $Gr(r, W)$.
$$\{ \lambda \in Gr(r, V) \mid W \subset \lambda \} \subset Gr(r, V)$$
is isomorphic to $Gr(r-m, V/W)$.
via the map
$$\begin{array}{ccc} \lambda & \longmapsto & \bar{\lambda} \\ \cap & & \cap \\ V & \longrightarrow & V/W. \end{array}$$
$\bar{\lambda}$ = Image under quotient map

· $Gr(r,V) = \{$ Space of projective linear spaces of dim $(r-1)$ in $\mathbb{P}V\}$.

An application - Lines on cubics

<u>Thm</u>: Every cubic hypersurface in $\mathbb{P}^3$ contains a line

<u>Pf</u>: Let $V = k[X,Y,Z,W]_3$.

$$= k\langle X^3, X^2Y, XY^2, Y^3, X^2Z, \ldots \rangle$$

$$\cong k^{\binom{3+3}{3}} = k^{20}.$$

Let $\mathbb{P} = \mathbb{P}V$. We think of $\mathbb{P}$ as the space of all cubic hypersurfaces. The point $[a_I]$ corresponds to $V(\sum a_I X^I)$. (I a multi index of deg 3).

Let $G = Gr(2,4)$
$= $ Space of projective lines in $\mathbb{P}^3$.

Consider $\Sigma = \{ (F,L) \mid F \in \mathbb{P}, L \in G, F|_L \equiv 0 \}$.

Claim 1: Σ is a closed subvar of $\mathbb{P} \times G$.

Proof: Let us check on charts.

For a multi index I_0, we have the standard affine open $\{[a_I] \mid a_{I_0} = 1\}$ of $\mathbb{P}$

For a 2-elt. subset $J_0 \subset \{1,2,3,4\}$ we have the std affine open $\{[M] \mid M_{J_0 \times 2} = \mathrm{id}\}$ of G. In the product, a point

$([a_I], [M]) \in \Sigma$ iff

$$\sum a_I (SM_1 + TM_2)^I \equiv 0. \quad — (*)$$

Note that the LHS is a cubic homog. poly in S, T with coeff which are poly. in the entries of M & a_I's. So $(*)$ is a system of poly. in the coordinates of the chart. So Σ is closed.

Claim 2: Σ is irred of dim 19.

Pf: Consider $\Sigma \xrightarrow{\pi} G$. We claim that for any $L \in G$, the fiber $\pi^{-1}(L)$ is a copy of $\mathbb{P}^{15}$. Indeed, $[a_I] \in \pi^{-1}(L)$ iff

$$\sum a_I x^I \big|_L \equiv 0.$$

Choose a param. $\mathbb{P}^1_{[S:T]} \xrightarrow{\sim} L$.

Then we have a restriction map

$$k[X,Y,Z,W]_3 \xrightarrow{r} k[S,T]_3.$$

& $\pi^{-1}(L) = \mathbb{P}\,\mathrm{Ker}(r)$.

See that r is surjective. The easiest proof is to first observe that the restriction

$$k[X,Y,Z,W]_1 \xrightarrow{r} k[S,T]_1$$

is surjective, so $S = r(\lambda_1)$
$T = r(\lambda_2)$

for some linear forms $\lambda_1, \lambda_2 \in \langle X,Y,Z,W \rangle$.

Then $S^3 = r(\lambda_1^3)$ $\quad S^2T = r(\lambda_1^2\lambda_2)$
$T^3 = r(\lambda_2^3)$ $\quad ST^2 = r(\lambda_1\lambda_2^2)$

$\Rightarrow$ $r : k[X,Y,Z,W]_3 \to k[S,T]_3$ is surj.

So $\mathrm{Ker}(r) \cong k^{16}$ $(20-4=16)$.

$\Rightarrow$ $\pi^{-1}(L) \cong \mathbb{P}^{15}$.

Since all fibers of π are irred. & 15 dim,

Σ is irred. & of dim $= 15 + \dim G$
$= 19$.

Claim 3: $\Sigma \xrightarrow{\mu} \mathbb{P}V$ is surj.

(i.e. every cubic contains a line).

Pf: $\dim \Sigma = 19$
$\dim \mathbb{P}V = 19$.

Check :- There exist a $p \in \mathbb{P}V$ s.t.
$\dim \mu^{-1}(p) = 0$.
For e.g. $p = [\text{Fermat cubic } X^3+Y^3+Z^3+W^3]$
Explicitly verify that it contains only the 27 lines & no more. (skipped).

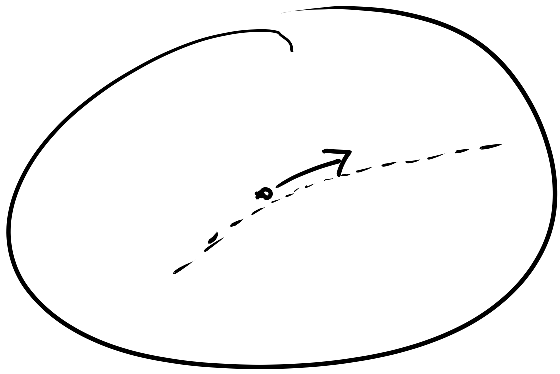
Since μ has a 0 dim fiber,
$\dim \mu(\Sigma) = \dim \Sigma = 19$, by the theorem on dim of fibers.
(if $\dim \mu(\Sigma) < 19$, then every fiber would have $\dim \geq \dim \Sigma - \dim \mu(\Sigma) \geq 1$.).

Since $\mu(\Sigma) \subset \mathbb{P}V$ is closed & $\dim \mu(\Sigma) = \dim \mathbb{P}V$, we must have $\mu(\Sigma) = \mathbb{P}V$ $\square$.

The Zariski Tangent Space

Motivation - Let X be a space.

A tangent vector on X is an "infinitesimal curve" on X.



In modern calculus / diff. geometry, we do not make this the literal definition, but in algebraic geometry, we can!

For simplicity, let X be affine, say $X \subset \mathbb{A}^n$. Set $R = k[X]$.

A point on X = max ideal of R

$$\begin{array}{ccc} \text{A point on } X & = & \text{max ideal of } R \\ \parallel & & \parallel \\ p : \bullet \to X & = & R \to k \end{array}$$

A curve, say $\mathbb{A}^1$, on X

$$\begin{array}{ccc} \text{A curve, say } \mathbb{A}^1\text{, on } X & & \\ \parallel & & \\ \gamma : \mathbb{A}^1 \to X & = & R \to k[t]. \end{array}$$

Def: A tangent vector on X is a map $R \to k[t]/t^2$

Geometry	Algebra
$\bullet \to X$	$R \to k$
$\mathbb{A}^1 \to X$	$R \to k[t]$
$? \to X$	$R \to k[t]/t^2$.

$?$ does not exist in the land of varieties, but it does in the land of "schemes".
$?$ should be thought as an "infinitesimal curve" $? = \bullet\!\nearrow$

If this sounds too fanciful, don't worry. We just take the RHS as the <u>definition</u>.

<u>One</u> <u>obs</u>: A tangent vector γ gives a point $\gamma(0)$:

$$\begin{tikzcd} R \arrow[r, "\gamma"] \arrow[dr, "\gamma(0)"'] & k[t]/t^2 \arrow[d] \\ & k \end{tikzcd}$$

$\gamma(0) :=$ Basepoint of γ.

Ex. $X = \mathbb{A}^n$.

$k[X] = k[x_1, \ldots, x_n]$.

$\gamma : k[x_1, \ldots, x_n] \mapsto k[t]/t^2$

$x_i \mapsto a_i + t b_i$

$\gamma(0) = (x_i \mapsto a_i)$ i.e. the point $(a_1, \ldots, a_n)$.

Def: Let $p \in X$. The tangent space to X at p is

$$T_p(X) = \{ \gamma \mid \gamma \text{ a tangent vector on } X \text{ with basepoint } p \}.$$

Ex. $T_{(a_1, \ldots, a_n)} \mathbb{A}^n \cong \{ (b_1, \ldots, b_n) \mid b_i \in k \}$
$\cong k^n$.

Ex: $X = V(y^2 - x) \subset \mathbb{A}^2$.

$R = k[x,y]/(y^2 - x)$

$$T_0 X = \left\{ \begin{array}{l} x \mapsto 0 + \alpha t \\ y \mapsto 0 + \beta t \end{array} \;\middle|\; \begin{array}{c} (\beta t)^2 - (\alpha t) \\ = 0 \text{ in } \\ k[t]/t^2 \end{array} \right\}$$

$\cong \{ \alpha \in k \} \cong k^1$.

More generally:

$X = V(f) \subset \mathbb{A}^n \qquad k[X] = k[x_1, \dots, x_n]/f$

$p = (a_1, \dots, a_n) \in X.$

$T_p X = \{ x_i \mapsto a_i + b_i t \mid f(a_i + b_i t) = 0 \text{ in } k[t]/t^2 \}.$

But $f(a_1 + b_1 t, \dots, a_n + b_n t)$

$$= f(a_1, \dots, a_n) + \sum b_i \frac{\partial f}{\partial x_i}(a_1, \dots, a_n) \cdot t$$

in $k[t]/t^2$.

& $f(a_1, \dots, a_n) = 0$ already.

So the only condition is:

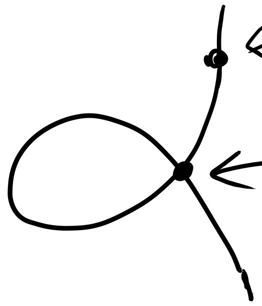
$$b_1 \frac{\partial f}{\partial x_1}(a_1, \dots, a_n) + \dots + b_n \frac{\partial f}{\partial x_n}(a_1, \dots a_n) = 0$$

So $T_p X = \{ (b_1, \dots, b_n) \mid \sum b_i \frac{\partial f}{\partial x_i}(a) = 0 \}.$

$\subset k^n$ vector subspace.

Obs:

$$\dim T_pX = \begin{cases} n-1 = \dim X & \text{if at least one partial non-zero} \\ n > \dim X & \text{if all partials zero} \end{cases}$$



X

More generally.

$I(X) = (f_1, \dots, f_\ell) \quad a \in X.$

$$T_pX = \left\{ x_i \mapsto a_i + b_i t \;\middle|\; \sum_i b_i \frac{\partial f_j}{\partial x_i}(a) = 0 \;\; \forall j = 1, \dots, \ell \right\}$$

$$\cong \left\{ (b_1, \dots, b_n) \;\middle|\; Jf(a) \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = 0 \right\} \subset k^n$$

$$(Jf)_{ij} = \frac{\partial f_j}{\partial x_i}$$

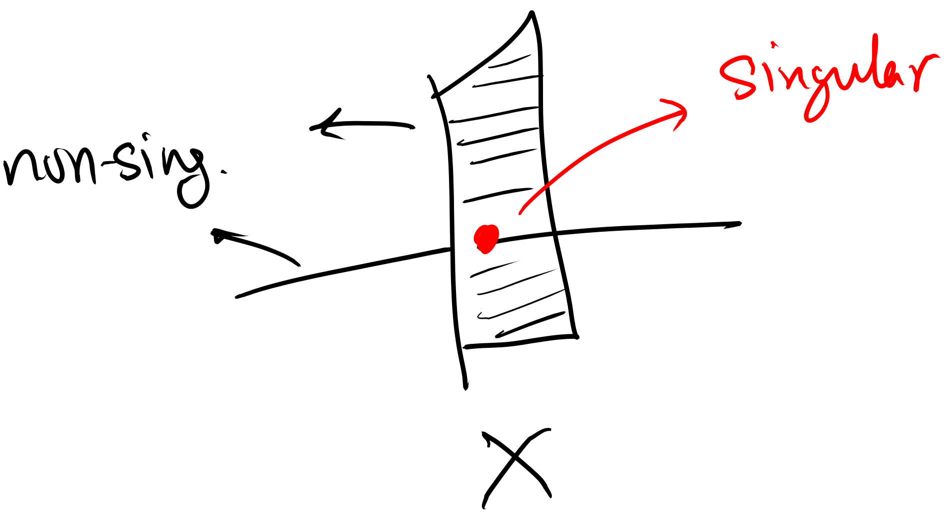
vector subspace.

Thm: We always have

$$\dim T_pX \geq \dim_p X$$

$\dim_p X$ = max dim of the comp. that contain p.

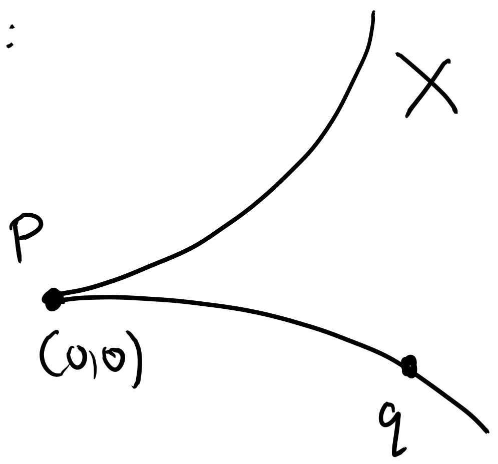
If equality holds, we say that X is non-singular or smooth at p.



Will not prove thm in general. But clear for hypersurfaces.

For general (non-affine X) — work on affine charts.

Examples:

① 

$y^2 - x^3$

$T_p X \cong k^2$.

$T_q X \cong k$ for all $q \neq (0,0)$.

② $C : V(Y^3 Z^2 - X^5)$

$(y^3 - x^5)$; $(z^2 - x^5)$.

Jf: $(-5x^4, 3y^2)$

$= (0,0)$ for $(x,y) = (0,0)$ in char > 5

for $x = 0$ in char 3

$y = 0$ in char 5

But only sing. is $(0,0) = [0:0:1]$

In the other chart, only need to check $(0,0) = [0:1:0]$ & this is a sing pt.

So $\text{sing}(C) = \{ [0:0:1], [0:1:0] \}$.