ALGEBRAIC GEOMETRY: WORKSHOP 2

1. How to do the last problem on HW1 for arbitrary k?

For simplicity, let us take $n = 2$, and recall the proof for $k = \mathbb{C}$. Let $D \subset \mathbb{A}^{2\times 2}$ be the set of diagonalizable matrices and $B \subset \mathbb{A}^{2\times 2}$ its complement. We show that neither D nor B are closed. To show that D is not closed, consider the family of matrices

$$M_t = \begin{pmatrix} 1 & 1 \\ 0 & t \end{pmatrix}.$$

See that $M_t \in D$ for $t \neq 1$, but $M_1 = \lim_{t\to 1} M_t \notin D$, which shows that D is not closed in the Euclidean topology, and hence also not in the Zariski topology. Similarly, by considering

$$N_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

we see that B is not closed.

We mimic the same proof algebraically. Instead of limits, we use more basic topology (remember the Zariski topology is not Hausdorff!). The role of the family M_t parametrised by $t \in \mathbb{R}$ is played by a similar family M_t parametrised by $t \in \mathbb{A}^1$.

Consider the map $M\colon\ \mathbb{A}^1 \to \mathbb{A}^{2\times 2}$ given by

$$M\colon t \mapsto \begin{pmatrix} 1 & 1 \\ 0 & t \end{pmatrix}.$$

Since M is defined by polynomial functions, it is contiuous in the Zariski topology. Note that $M^{-1}(D) = \mathbb{A}^1 \setminus \{1\}$ is not Zariski closed. Therefore, D is not Zariski closed. Similarly, we show that B is not Zariski closed.

2. More exercises with ideals and their vanishing loci.

(1) Let k be an algebraically closed field of characteristic not equal to 2. For $c \in k$, let Z_c be the algebraic subset of $\mathbb{A}^2_k$ defined by $x^2 + y^2 = 1$ and $x = c$. Find $I(Z_c)$ for all values of $c \in k$ (Caution: Pay close attention to two special values of c).

(2) Draw a picture of the special and the general situation by taking $k = \mathbb{R}$

(3) What happens if the characteristic of k is 2?

3. The Zariski topology is not Hausdorff.

Let k be an algebraically closed field. Let us show that the Zariski topology on $\mathbb{A}^n_k$ is not Hausdorff. In fact, let us show that any two non-empty subset of $\mathbb{A}^n_k$ have a non-empty intersection.

(1) For $n = 1$, recall that the Zariski topology is the finite complement topology, and conclude.

(2) In general, show that every Zariski open $U \subset \mathbb{A}^n_k$ contains a *basic open*, namely an open set of the form

$$D(f) = \{x \mid f(x) \neq 0\}.$$

(3) Show that $D(f) \cap D(g) = D(fg)$, and conclude that any two non-empty opens must intersect.