ALGEBRAIC GEOMETRY: WORKSHOP 3

Let us explore some important finiteness properties of the Zariski topology on affine varieties.

1. Compactness

Let X be an affine variety. We will show that X is compact in the Zariski topology (every open cover has a finite subcover). The ideas involved are very close to the ideas we used in Friday's class.

(1) A *basic open set* of X is a set of the form

$$D_f = \{x \in X \mid f(x) \neq 0\},$$

where $f \in k[X]$. Show that every open subset of X is a union of basic open sets.

(2) Use the Nullstellensatz to show that an open cover of X by basic open sets has a finite subcover.

(3) Conclude that every open cover of X has a finite subcover.

2. Noetherian-ness

A *Noetherian topological space* is one where every descending chain of closed subsets stabilizes. Prove that every affine variety is a Noetherian topological space.

3. The punctured line

Now a concrete problem. Let $X = \mathbb{A}^1 \setminus \{0, 1\}$. Describe the ring of regular functions $k[X]$ as a subring of the field $k(x)$. Can you generalise your result to $X = \mathbb{A}^1 \setminus \{a_1, \ldots, a_n\}$?