ALGEBRAIC GEOMETRY: WORKSHOP 7

In class, we showed that a closed subset of $\mathbb{P}^n \times \mathbb{A}^m$ projects down to a closed subset of $\mathbb{A}^m$. The proof, however, did not give a good way of computing this closed subset. Today, we will see how to find the equations for this set for $n = 1$ and when $Z \subset \mathbb{P}^1 \times \mathbb{A}^m$ is defined by two polynomials.

The basic question is the following. Consider a system

$$F(X, Y) = 0 \text{ and } G(X, Y) = 0,$$

where F and G are homogeneous of degrees d and e, respectively. When does the system have a non-zero solution (X, Y)?

(1) Show that the system above has a non-zero solution if an only if F and G have at least one common linear factor.

(2) Show that F and G have a common linear factor if and only if there exists a homogeneous polynomial A of degree $e - 1$ and a homogeneous polynomial B of degree $d - 1$ such that

$$AF + BG = 0.$$

(3) Show that the existence of A and B as above is equivalent to the non-injectivity of the following linear map

$$m\colon k[X, Y]_{e-1} \oplus k[X, Y]_{d-1} \to k[X, Y]_{d+e-1}$$

defined by

$$m\colon (A, B) \mapsto AF + BG.$$

(4) Let $F(X, Y, s, t) = sX^2 + tY^2$ and $G(X, Y, s, t) = X^2 + stXY + Y^2$. Using the previous part, write an equation in s, t that is satisfied precisely when $F(X, Y, s, t)$ and $G(X, Y, s, t)$ have a common zero in $\mathbb{P}^1$. Your equation will have the form $\det \cdots = 0$. The matrix $\cdots$ is called the *resultant* matrix of F and G and its determinant is called the *resultant*.

(5) The general case is similar. Given $F(X, Y, t_1, \ldots, t_m)$ and $G(X, Y, t_1, \ldots, t_m)$, the projection of $V(F, G) \subset \mathbb{P}^1 \times \mathbb{A}^m$ in $\mathbb{A}^m$ is cut out by the resultant of F and G.