ALGEBRAIC GEOMETRY: WORKSHOP 8

Let us prove the following result asserted in class.

Proposition 1. *A projective variety $X \subset \mathbb{P}^n$ is irreducible if and only if the cone $CX \subset \mathbb{A}^{n+1}$ is irreducible.*

Fo the proof, let us import the notion of a fiber bundle from geometry/topology. Let X and F be varieties.

Definition 2. A *fiber bundle* over X with fiber F is a variety Y along with a map $\pi\colon Y \to X$ such that the following holds: there exists an open covering U_i of X and for every i an isomorphism $\pi^{-1}U_i \cong F \times U_i$, compatible with the projection to U_i.

For example, $Y = X \times F \to X$ is a fiber bundle with fiber F. Let us abbreviate "fiber bundle with fiber F" by "F-bundle".

(1) Show that $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is an $\mathbb{A}^1 \setminus \{0\}$ bundle.
(2) Show that if $\pi\colon Y \to X$ is an F-bundle and $Z \subset X$ is a open/closed subvariety, then $\pi^{-1}Z \to Z$ is also an F-bundle.
(3) Adapt the proof from class to show that if F and X are irreducible and $Y \to X$ is an F-bundle, then Y is also irreducible.
(4) Conclude that $X \subset \mathbb{P}^n$ is irreducible if and only if $CX \setminus \{0\}$ is irreducible.
(5) Conclude that $X \subset \mathbb{P}^n$ is irreducible if and only if CX is irreducible.

For Luke:

The proof of the third part I have in mind is the following. Let $\pi\colon Y \to X$ be an F-bundle. Suppose $Y = A \cup B$ where A and B are closed subsets of Y. We want to prove that $A = Y$ or $B = Y$.

For every $x \in X$, the fiber Y_x is a copy of F, which is irreducible. So $A_x = Y_x$ or $B_x = Y_x$. Let $\alpha \subset X$ (resp. β) be the set of $x \in X$ such that $A_x = Y_x$ (resp. $B_x = Y_x$).

We proved in class that for a product (i.e. trivial bundle), α and β are closed subsets of X. More generally, the same holds. From the result in class, it follows that $U_i \cap \alpha$ and $U_i \cap \beta$ are closed in U_i, and since the U_i form an open cover of X, we see that α and β are closed in X.

Finally, since $\alpha \cup \beta = X$, from the irreducibility of X, we see that $\alpha = X$ or $\beta = X$. So $A = Y$ or $B = Y$.