ALGEBRAIC GEOMETRY: WORKSHOP 9

The Blowup

Let $\Gamma \subset \mathbb{A}^2 \setminus (0,0) \times \mathbb{P}^1$ be the graph of the rational map

$$(x, y) \mapsto [x : y],$$

and let $B = \overline{\Gamma}$ be its closure.

(1) Show that B is given by the equation

$$B = \{(x, y, [u : v]) \mid uv - yu = 0\}.$$

(2) Describe the two affine charts of B obtained by taking $u \neq 0$ and $v \neq 0$, and describe the transition functions.
(3) Show that B is irreducible of dimension 2.
(4) Study the maps $B \to \mathbb{A}^2$ and $B \to \mathbb{P}^1$, defined by the two projections. Show that the first map is an isomorphism over $\mathbb{A}^2 \setminus (0,0)$, and its fiber over $(0,0)$ is a copy of $\mathbb{P}^1$. Show that the second map has fibers isomorphic to $\mathbb{A}^1$, and the images of these fibers to $\mathbb{A}^2$ are lines through the origin.
(5) The fiber of $B \to \mathbb{A}^2$ over $(0,0)$ is called the *exceptional divisor*. Show that the points of the exceptional divisor are in a natural bijection with the lines in $\mathbb{A}^2$ through $(0,0)$.

To summarize: B is a birational modification of $\mathbb{A}^2$ obtained by replacing the origin by the set of lines through the origin. Thanks to this surgery, the rational map $\mathbb{A}^2 \dashrightarrow \mathbb{P}^1$ given by $(x, y) \mapsto [x : y]$ has been resolved to a regular map $B \to \mathbb{P}^1$. We say that B is obtained from $\mathbb{A}^2$ by *blowing up* the origin. [1]

(6) Taking the base field to be $\mathbb{R}$, try to draw a picture of B.
(7) Do the analogous thing for $\mathbb{A}^n$ with the map $\mathbb{A}^n \dashrightarrow \mathbb{P}^{n-1}$ given by $(x_1, \ldots, x_n) \mapsto [x_1 : \cdots : x_n]$.

[1] I do not know the etymology of this term.