ALGEBRAIC GEOMETRY: WORKSHOP 10

The Grassmannian Gr(2, 4)

By definition, points of $\mathrm{Gr}(2,4)$ correspond to 2 dimensional subspaces $V \subset k^4$. By choosing a basis, we may represent V as the column span of a 2×4 matrix M. Given V, the matrix M is unique up to the action of GL_2 by right multiplication.

(1) Consider the Plücker map

$$\mathrm{Gr}(2,4) \longrightarrow \mathbb{P}^5 = \{[U_{12} : U_{13} : U_{14} : U_{23} : U_{24} : U_{34}]\}.$$

Let $W_{12} \subset \mathrm{Gr}(2,4)$ be the preimage of $\{U_{12} \neq 0\}$. Identify W_{12} with $\mathbb{A}^4$ by making the first 2×2 block of M equal to the identity. Identify $\{U_{12} \neq 0\}$ with $\mathbb{A}^5$ in the standard way, by making $U_{12} = 1$. Write down the Plücker map

$$W_{12} = \mathbb{A}^4 \longrightarrow \mathbb{A}^5 = \{U_{12} \neq 0\}.$$

See that the image is a closed subset, and the map is an isomorphism onto it.

(2) Show that the image of $\mathrm{Gr}(2,4) \subset \mathbb{P}^5$ is a degree 2 hypersurface.

(3) Fix a "flag" in k^4, namely vector spaces $V_1 \subset V_2 \subset V_3$ of dimensions 1, 2, 3, respectively. By projectivising everything, we may view $\mathrm{Gr}(2,4)$ as the space of (projective) lines in $\mathbb{P}^3$. Then a flag corresponds to $p \in L \subset P$, where p is a point, L is a line, and P is a plane in $\mathbb{P}^3$. Up to coordinate changes on k^4, all flags are equivalent, so you may take the standard (coordinate) flag for explicit calculations. Show that the following are closed subsets of $\mathrm{Gr}(2,4)$:

- (a) $\sigma_0 = \mathrm{Gr}(2,4)$,
- (b) $\sigma_1 = \{V \mid V \cap V_2 \neq 0\} = \{\text{Lines meeting } L\}$,
- (c) $\sigma_2 = \{V \mid V_1 \subset V\} = \{\text{Lines through } p\}$,
- (d) $\sigma_{11} = \{V \mid V \subset V_3\} = \{\text{Lines in } P\}$,
- (e) $\sigma_{21} = \{V \mid V_1 \subset V \subset V_3\} = \{\text{Lines in } P \text{ through } p\}$,
- (f) $\sigma_{22} = \{V = V_2\} = \{L\}$.

One way to do this is to translate these conditions in terms of the matrix M.

(4) See that these six sets correspond to six possible echelon forms of M.

(5) Find the (co)-dimensions of these sets. Also see that

$$\sigma_2 \cong \mathbb{P}^2, \quad \sigma_{11} \cong \mathbb{P}^2, \quad \sigma_{21} \cong \mathbb{P}^1.$$

(6) Draw the poset formed by the sets σ under inclusion.

In general, $\mathrm{Gr}(r,n)$ has a stratification by closed subsets indexed by Young tableaux that fit in an $r \times (n-r)$ box. These subsets are called "Schubert cells." The inclusion relations correspond to the dominance order.