MATH 465/565: Grassmannian Notes

The *Grassmannian* $G(r,n)$ is the set of r-dimensional subspaces of the k-vector space k^n; it has a natural bijection with the set $\mathbb{G}(r-1,n-1)$ of $(r-1)$-dimensional linear subspaces $\mathbb{P}^{r-1} \subseteq \mathbb{P}^n$. We write $G(k,V)$ for the set of k-dimensional subspaces of an n-dimensional k-vector space V.

We'd like to be able to think of $G(r,V)$ as a quasiprojective variety; to do so, we consider the *Plücker embedding*:

$$\gamma\colon G(r,V) \to \mathbb{P}\Big(\bigwedge^r V\Big)$$
$$\mathrm{Span}(v_1,\ldots,v_r) \mapsto [v_1 \wedge \cdots \wedge v_r]$$

If $\left(w_i = \sum_j a_{ij} v_j\right)_{1\le i\le r}$ is another ordered basis for $\Lambda = \mathrm{Span}(v_1,\ldots,v_r)$, where $A = (a_{ij})$ is an invertible matrix, then $w_1 \wedge \cdots \wedge w_r = (\det A)(v_1 \wedge \cdots \wedge v_r)$. Thus the Plücker embedding is a well-defined function from $G(k,V)$ to $\mathbb{P}\big(\bigwedge^r V\big)$. We would like to show, in analogy with what we were able to show for the Segre embedding $\sigma\colon \mathbb{P}(V)\times\mathbb{P}(W) \to \mathbb{P}(V\otimes W)$, that

- the Plücker embedding γ is injective,
- the image $\gamma(G(r,V))$ is closed, and
- the Grassmannian $G(r,V)$ "locally" can be given a structure as an affine variety, and γ restricts to an isomorphism between these "local" pieces of $G(r,V)$ and Zariski open subsets of the image.

Given $x \in \bigwedge^r V$, we say that x is *totally decomposable* if $x = v_1 \wedge \cdots \wedge v_r$ for some $v_1,\ldots,v_r \in V$, or equivalently, if $[x]$ is in the image of the Plücker embedding.

Example. Every non-zero element of $\bigwedge^1 V$ is trivially totally decomposable.

Example. If $\dim V = 3$, then every non-zero element of $\bigwedge^2 V$ is totally decomposable.

Proof. Given a sum $v_1\wedge v_2 + v_3 \wedge v_4$ of two non-zero elements of $\bigwedge^2 V$, the two-dimensional subspaces $\mathrm{Span}(v_1,v_2)$ and $\mathrm{Span}(v_3,v_4)$ must intersect. If w_1 is in the intersection, then we can rewrite $v_1\wedge v_2 = w_1 \wedge w_2$ for some $w_1 \in \mathrm{Span}(v_1,v_2)$. Similarly we can rewrite $v_3\wedge v_4 = w_1\wedge w_3$ for some w_3. Then

$$v_1\wedge v_2 + v_3\wedge v_4 = w_1\wedge w_2 + w_1\wedge w_3 = w_1\wedge(w_2+w_3)$$

is totally decomposable. Proceeding in the same way by induction, we can show that any element of $\bigwedge^2 V$ is totally decomposable. □

Example. On the other hand, if $v_1,v_2,v_3,v_4 \in V$ are linearly independent, then $v_1\wedge v_2 + v_3\wedge v_4$ is not totally decomposable. This follows (for $\mathrm{char} k \neq 0$) from the observation that since $v\wedge v = 0$, if $x \in \bigwedge^r V$ is totally decomposable, then $x \wedge x = 0$. Since

$$\begin{aligned}(v_1\wedge v_2 + v_3\wedge v_4)\wedge(v_1\wedge v_2 + v_3\wedge v_4) &= v_1\wedge v_2\wedge v_3\wedge v_4 + v_3\wedge v_4\wedge v_1\wedge v_2\\ &= 2v_1\wedge v_2\wedge v_3\wedge v_4,\end{aligned}$$

which is non-zero since $v_1,v_2,v_3,v_4 \in V$ are linearly independent (certainly $v_3\wedge v_4\wedge v_1\wedge v_2 = \pm v_1\wedge v_2\wedge v_3\wedge v_4$, and the sign is positive because $(13)(24)$ is an even permutation).

If $e_1, \ldots, e_n$ is a basis for V, then the set of $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}$ is a basis for $\bigwedge^r V$, where $I = \{i_1, \ldots, i_r\}$, with $1 \le i_1 < i_2 < \cdots < i_r \le n$. Thus any element of $x \in \bigwedge^r V$ has a unique representation in the form

$$x = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I|=r}} a_I e_I = \sum_{1 \le i_1 < \cdots < i_r \le n} a_{i_1,i_2,\ldots,i_r} (e_{i_1} \wedge \cdots \wedge e_{i_r}),$$

and we call the homogeneous coordinates a_I the *Plücker coordinates* on $\mathbb{P}\big(\bigwedge^r V\big) \cong \mathbb{P}^{\binom{n}{r}-1}$ associated to the choice of ordered basis $(e_1, \ldots, e_n)$ for V.

Example. If V has basis e_1, e_2, e_3, e_4, then every element $x \in \bigwedge^2 V$ can be uniquely written as

$$x = a_{12}(e_1 \wedge e_2) + a_{13}(e_1 \wedge e_3) + a_{14}(e_1 \wedge e_4) + a_{23}(e_2 \wedge e_3) + a_{24}(e_2 \wedge e_4) + a_{34}(e_3 \wedge e_4).$$

If x is totally decomposable, then we know $x \wedge x = 0$; we compute in terms of the Plücker coordinates that

$$x \wedge x = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})(e_1 \wedge e_2 \wedge e_3 \wedge e_4),$$

Hence the image of the Plücker embedding of $G(2,4)$ into $\mathbb{P}^5$ satisfies the homogeneous quadric equation $a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0$. (In fact, in this particular case the image is precisely the zero locus of this polynomial.)

To show that the Plücker embedding is an injection, we must describe how to recover $\mathrm{Span}(v_1, \ldots, v_r)$ from $x = v_1 \wedge \cdots \wedge v_r$. We observe first that $v_i \wedge x = 0$ for $1 \le i \le r$, and more generally, if $v \in \mathrm{Span}(v_1, \ldots, v_r)$, then $v \wedge x$ is zero. In fact, this property determines $\mathrm{Span}(v_1, \ldots, v_r)$:

Proposition. *Given a non-zero $x \in \bigwedge^r V$, let $\varphi_x \colon V \to \bigwedge^{r+1} V$ be the linear map*

$$\varphi_x(v) = v \wedge x.$$

Then $\dim\ker(\varphi_x) \le r$, with equality if and only if x is totally decomposable. If $x = v_1 \wedge \cdots \wedge v_r$, then $\ker(\varphi_x) = \mathrm{Span}(v_1, \ldots, v_r)$.

Proof. Chose a basis $e_1, \ldots, e_n$ for V so that $e_1, \ldots, e_s$ is a basis for $\ker(\varphi_x)$, where $s = \dim\ker(\varphi_x)$. Let $x = \sum_{|I|=r} a_I e_I$. Then we have

$$\varphi_x(e_i) = e_i \wedge x = \sum_{|I|=r} a_I (e_i \wedge e_I) = \sum_{|I|=r,\ I \not\ni i} \pm a_I e_{I \cup \{i\}},$$

so if $e_i \wedge x = 0$, then $a_I = 0$ whenever $i \notin I$, or equivalently, every non-zero term of x involves e_i. Since this is true for every i with $1 \le i \le s$, we have $s \le r$ and we can write $x = (e_1 \wedge \cdots \wedge e_s) \wedge y$ for some $y \in \bigwedge^{r-s} V$. In the case that $r = s$, we get that $x = a_{1,\ldots,r} e_1 \wedge \cdots \wedge e_r$, and x is totally decomposable.

As for the second statement, in the case $x = v_1 \wedge \cdots \wedge v_r$, we know that $\mathrm{Span}(v_1, \ldots, v_r) \subseteq \ker(\varphi_x)$, and we have just shown that both spaces have the same dimension. □

Corollary. *The Plücker embedding is an injection and its image is closed in $\mathbb{P}\big(\bigwedge^k V\big)$*

Proof. Given $x = v_1 \wedge \cdots \wedge v_r$, we've shown $\mathrm{Span}(v_1, \ldots, v_r) = \ker(\varphi_x)$, so we can recover the r-dimensional subspace $\mathrm{Span}(v_1, \ldots, v_r)$ from x, and the Plücker embedding is injective.

To show that its image is closed, we note that since $\dim \ker(\varphi_x) \leq r$, with equality if and only if x is totally decomposable, $\mathrm{Rank}(\varphi_x) \geq n - r$, with equality if and only if x is totally decomposable. If M_x is a matrix for φ_x in a given basis $e_1, \ldots e_n$ for V and the corresponding basis for $\bigwedge^{r+1} V$, with $x = \sum_{|I|=r} a_I e_I$, then the entries of M_x are all 0 or $\pm a_I$.

The condition that φ_x have rank at most $n-r$ is expressed by the vanishing of all the $(n-r+1) \times (n-r+1)$ minors of the matrix M_x. Since the entries of M_x are homogeneous linear in the Plücker coordinates a_I, we find that the image of the Plücker embedding is the set of common zeros of a collection of homogeneous polynomials of degree $n-r+1$ in the a_I. Thus the image is closed. □

Example. We return to the case of $G(2,4)$, with

$$x = a_{12}(e_1 \wedge e_2) + a_{13}(e_1 \wedge e_3) + a_{14}(e_1 \wedge e_4) + a_{23}(e_2 \wedge e_3) + a_{24}(e_2 \wedge e_4) + a_{34}(e_3 \wedge e_4).$$

In the bases (e_1, e_2, e_3, e_4) for V and $(e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4)$ for $\bigwedge^3 V$, the matrix of φ_x is

$$M_x = \begin{bmatrix} a_{23} & -a_{13} & a_{12} & 0 \\ a_{24} & -a_{14} & 0 & a_{12} \\ a_{34} & 0 & -a_{14} & a_{13} \\ 0 & a_{34} & -a_{24} & a_{24} \end{bmatrix}.$$

Here, it's easy to see directly that if any entry of this matrix is non-zero, the matrix must have rank at least 2. The image of $G(2,4)$ under the Plücker embedding is the common zero locus of the sixteen 3×3 minors of this matrix, such as

$$\begin{vmatrix} a_{23} & -a_{13} & a_{12} \\ a_{24} & -a_{14} & 0 \\ a_{34} & 0 & -a_{14} \end{vmatrix} = a_{23}(-a_{14})(-a_{14}) - a_{24}(-a_{13})(-a_{14}) - a_{34}(-a_{14})a_{12} = a_{14}(a_{14}a_{23} - a_{13}a_{24} + a_{12}a_{34}),$$

$$\begin{vmatrix} a_{23} & -a_{13} & a_{12} \\ a_{24} & -a_{14} & 0 \\ 0 & a_{34} & -a_{24} \end{vmatrix} = a_{23}(-a_{14})(-a_{24}) + a_{12}a_{24}a_{34} - a_{24}(-a_{13})(-a_{24}) = a_{24}(a_{14}a_{23} + a_{12}a_{34} - a_{13}a_{24}), \text{ and}$$

$$\begin{vmatrix} -a_{13} & a_{12} & 0 \\ -a_{14} & 0 & a_{12} \\ 0 & -a_{14} & a_{13} \end{vmatrix} = -(-a_{13})(a_{12})(-a_{14}) - a_{12}(-a_{14})a_{13} = 0.$$

Indeed, four of the 3×3 minors are zero, and the rest are all multiples of the degree two polynomial $a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$ that we found earlier by $\pm a_I$. In general, while the equations we found for the Plücker image have degree $n-r+1$, in fact in characteristic 0 the homogeneous ideal of the Plücker image is always generated by degree 2 polynomials.

Local coordinates on Grassmannians

Given an $r \times n$ matrix $B = (b_{ij})$ of rank r, the row space of B maps under the Plücker embedding to

$$(b_{11}e_1 + \cdots + b_{1n}e_n) \wedge \cdots \wedge (b_{r1}e_1 + \cdots + b_{rn}e_n) = \sum_{|J|=r} a_J e_J,$$

where the a_I are the usual Plücker coordinates. In this product, only the terms $b_{ij}e_j$ with $j \in J$ contribute to the term $a_J e_J$, and

$$(b_{1j_1}e_{j_1} + \cdots + b_{1j_r}e_{j_r}) \wedge \cdots \wedge (b_{rj_1}e_{j_1} + \cdots + b_{rj_n}e_{j_n}) = (\det(b_{ij_l})_{1\le i,l\le r})(e_{j_1} \wedge \cdots \wedge e_{j_r}),$$

i.e. the Plücker coordinate a_J is the $r \times r$ minor of the matrix B obtained by taking all r rows and the r columns with indices in J.

We wish to describe the open subsets of $G(r,n)$ where some $a_J \neq 0$. For simplicity of notation, we will consider the case where $a_{1,\dots,r} \neq 0$; every other case is equivalent to this one by permuting our basis for V. The corresponding minor of B is just the determinant of the leftmost $r \times r$ submatrix, and condition $a_{1,\dots,r} \neq 0$ means this submatrix is invertible. Multiplying on the left by the inverse of this matrix, we can replace our matrix with a new matrix B of the form

$$B = \begin{bmatrix} 1 & 0 & \dots & 0 & b_{1,r+1} & b_{1,r+2} & \dots & b_{1,n} \\ 0 & 1 & \dots & 0 & b_{2,r+1} & b_{2,r+2} & \dots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & b_{r,r+1} & b_{r,r+2} & \dots & b_{r,n} \end{bmatrix}$$

with the same row space. Equivalently, performing elementary row operations on B changes basis for the row space (without changing the basis for V), and using Gaussian elimination we can put B in reduced row echelon form. The condition that the the leftmost $r \times r$ minor is non-zero implies that the leading 1 in each row will be in this position.

Moreover, any two distinct matrices of this form will have different row spaces. Thus we can think of these b_{ij} for $1 \le i \le r$ and $r+1 \le j \le n$ as "local coordinates" on $G(r,n)$, and the Plücker gives us a bijection between $\mathbb{A}^{r(n-r)}$ (with coordinates b_{ij}) and the open subset $a_{1,\dots,r} \neq 0$ of the Plücker image. The a_J are given by the $r \times r$ minors of this matrix, which are certainly polynomials in the entries, so to show that the Plücker embedding is "locally an isomorphism" we must show that the inverse is regular, i.e. we must compute the b_{ij} in terms a_J.

To do so, we must compute other minors of this $r \times n$ matrix B; for example,

$$a_{2,3,\dots,r,j} = \begin{vmatrix} 0 & \dots & 0 & b_{1,j} \\ 1 & \dots & 0 & b_{2,j} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & b_{r,j} \end{vmatrix} = (-1)^{r+1} b_{1,j},$$

as we can see by Gaussian elimination or by expanding by minors across the first row. Similarly, for this matrix B, we have $a_{1,\dots,\hat{i},\dots,r,j} = (-1)^{r+i} b_{i,j}$, where $\hat{i}$ means that the i is omitted. Of course, this was for matrices with the special form above, where in particular $a_{1,\dots,r} = 1$; to express the $b_{i,j}$ as regular functions in the a_J on the open subset $a_{1,\dots,r} \neq 0$ of the Plücker image, we homogenize, yielding

$$b_{i,j} = (-1)^{r+i} \frac{a_{1,\dots,\hat{i},\dots,r,j}}{a_{1,\dots,r}}.$$

Thus the Plücker embedding is "locally an isomorphism." Whenever we write $G(r,n)$, we will think of it as a quasiprojective variety by identifying it with its image under the Plücker embedding. On the other hand, to show for example that a map from $G(r,n)$ is regular or that a subset of $G(r,n)$ is closed, it suffices by what we've just shown to check these properties on each of the affine open sets (isomorphic to $\mathbb{A}^{r(n-r)}$) that we've described.

Remark. We can describe our local coordinate charts without reference to a basis. Given a subspace $\Gamma \subseteq V$ of dimension $n - r$, we can consider the set

$$U_\Gamma = \{\Lambda \in G(r, V): \ \Lambda \cap \Gamma = (0).\}$$

Then U_Γ is an open subset of $G(r, V)$, and if we fix a $\Lambda_0 \in U_\Gamma$, then every element of U_Γ has the form

$$\Lambda_\alpha = \{v + \alpha(v): \ v \in \Lambda_0\}$$

for a unique $\alpha \in \mathrm{Hom}_k(\Lambda_0, \Gamma)$. This gives a bijection $U_\Gamma \cong \mathrm{Hom}_k(\Lambda_0, \Gamma)$.

The Plücker relations

We saw above that the Plücker coordinates of the row space of

$$B = \begin{bmatrix} 1 & 0 & \dots & 0 & b_{1,r+1} & b_{1,r+2} & \dots & b_{1,n} \\ 0 & 1 & \dots & 0 & b_{2,r+1} & b_{2,r+2} & \dots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & b_{r,r+1} & b_{r,r+2} & \dots & b_{r,n} \end{bmatrix}$$

are given by $b_{i,j} = (-1)^{r+i} \frac{a_{1,\dots,\hat{i},\dots,r,j}}{a_{1,\dots,r}}$. Similarly, the $r \times r$ minor of this matrix including all but s of the first r columns, with the omitted columns indexed by $i_1, \dots, i_s$, and s of the remaining columns, indexed by $j_1, \dots, j_s$, will be (up to sign) equal to the $s \times s$ minor obtained by taking rows $i_1, \dots, i_s$ and columns $j_1, \dots, j_s$.

Taking $a_{1,\dots,r} = 1$, we find then that determinant of this $s \times s$ submatrix is, up to sign, equal to $a_{1,\dots,\hat{i_1},\dots,\hat{i_s},\dots,r,j_1,\dots,j_s}$. On the other hand, we could compute this determinant by expanding by minors along some row or column, giving an expression in terms of the $b_{i,j} = (-1)^{r+i} a_{1,\dots,\hat{i},\dots,r,j}$ multiplied by $(s-1) \times (s-1)$ minors, which in turn are, up to sign, equal to Plücker coordinates a_J. Computing this determinant in these two different ways, we can obtain a quadratic relation on the Plücker coordinates. The quadratic relations we obtain in this way are the *Plücker relations* involving $a_{1,\dots,r}$.

For example, for $a_{1,\dots,r} = 1$, we have

$$a_{3,\dots,r,j,l} = \begin{vmatrix} 0 & 0 & \dots & 0 & b_{1,j} & b_{1,l} \\ 0 & 0 & \dots & 0 & b_{2,j} & b_{2,l} \\ 1 & 0 & \dots & 0 & b_{3,j} & b_{2,l} \\ 0 & 1 & \dots & 0 & b_{4,j} & b_{2,l} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & b_{r,j} & b_{r,l} \end{vmatrix} = (-1)^r \begin{vmatrix} b_{1,j} & b_{1,l} \\ b_{2,j} & b_{2,l} \end{vmatrix},$$

but expanding this 2×2 determinant by minors, we can also compute (again, taking $a_{1,\dots,r} = 1$) that

$$\begin{aligned} \begin{vmatrix} b_{1,j} & b_{1,l} \\ b_{2,j} & b_{2,l} \end{vmatrix} &= b_{1,j} b_{2,l} - b_{1,l} b_{2,j} \\ &= (-1)^{r+1} a_{2,3,\dots,r,j} (-1)^{r+2} a_{1,3,\dots,r,l} - (-1)^{r+1} a_{2,3,\dots,r,l} (-1)^{r+2} a_{1,3,\dots,r,j} \\ &= a_{1,3,\dots,r,j} a_{2,3,\dots,r,l} - a_{1,3,\dots,r,l} a_{2,3,\dots,r,j}, \end{aligned}$$

which yields $(-1)^r a_{3,\dots,r,j,l} = a_{1,3,\dots,r,j} a_{2,3,\dots,r,l} - a_{1,3,\dots,r,l} a_{2,3,\dots,r,j}$. Homogenizing this relation, we get

$$(-1)^r a_{1,2,\dots,r} a_{3,\dots,r,j,l} = a_{1,3,\dots,r,j} a_{2,3,\dots,r,l} - a_{1,3,\dots,r,l} a_{2,3,\dots,r,j}.$$

Note that in the special case $n = 4$, $r = 2$, $j = 3$, and $l = 4$, we recover

$$a_{12} a_{34} = a_{13} a_{24} - a_{14} a_{23},$$

which we saw previously is the single equation for the image of $G(2,4)$ under the Plücker embedding.

Remark. See Shafarevich or Harris (handout on website) for an exposition of the Plücker relations that does not rely on local coordinates. The approach is essentially the same in both books, although the notation is different. Harris refers to a somewhat natural isomorphism $\bigwedge^r V \cong \bigwedge^{n-r} V^*$; to construct this isomorphism, note first that there is a non-degenerate bilinear map

$$\bigwedge^r V \times \bigwedge^{n-r} V \longrightarrow \bigwedge^n V,$$

defined by $(x, y) \mapsto x \wedge y$. The vector space $\bigwedge^n V$ is one-dimensional, so if we choose an identification $k \cong \bigwedge^n V$, this non-degenerate bilinear map induces an isomorphism $\bigwedge^r V \cong (\bigwedge^{n-r} V)^*$, natural only up to multiplication by a scalar because of the choice of a basis for $\bigwedge^n V$.

Now, there is a natural bilinear map

$$\bigwedge^{n-r} V^* \times \bigwedge^{n-r} V \longrightarrow k,$$

defined by $(u, x) \mapsto u \lrcorner x$, called the *convolution* of u and x (see Shafarevich for the definition). This induces a natural isomorphism $(\bigwedge^{n-r} V)^* \cong \bigwedge^{n-r} V^*$. Composing these two isomorphisms, we get an isomorphism

$$\bigwedge^r V \cong \left(\bigwedge^{n-r} V\right)^* \cong \bigwedge^{n-r} V^*,$$

natural up to multiplication by a scalar (due to the choice of identification $k \cong \bigwedge^n V$). For $x \in \bigwedge^r V$, Harris uses x^* to denote the corresponding element of $\bigwedge^{n-r} V^*$.