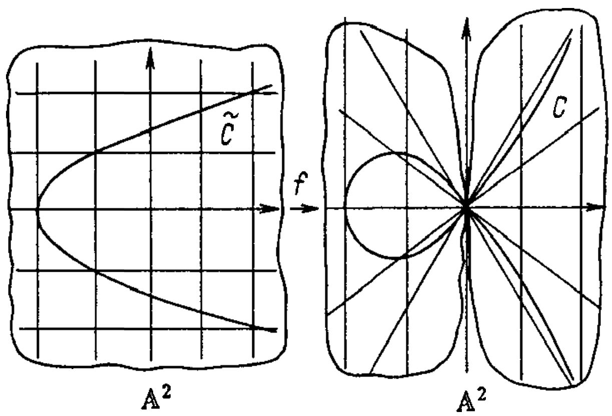


Fig. 4

compared with the ordinary metric topology on $\mathbb{C}^n$. Of course the polynomial functions on $\mathbb{C}^n$ are continuous in the topology given by the Euclidean metric. It follows that the classical topology on $\mathbb{C}^n$ is stronger than the Zariski topology. In other words, a Zariski open (respectively, closed) subset is also open (respectively, closed) in the classical topology. The converse is not true; for instance, in $\mathbb{C}$ the algebraic sets are exactly $\mathbb{C}$ and its finite subsets. Thus the Zariski open subsets are 'very big'; in particular, the Zariski topology is highly non-Hausdorff.

A further difference with the classical topology is that the Zariski topology on the product $X \times Y$ of two affine varieties is stronger than the product of the Zariski topologies on $X$ and $Y$. So, in $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$ there are many infinite algebraic subsets which are not made up of vertical and horizontal lines (for example, the diagonal).

Although there is quite a distance between the Zariski topology and the classical one, they are not divided by an impassable chasm. Here is the easiest footbridge joining them: if an open subset $U \subset X$ is Zariski dense, it is dense also in the classical topology. More subtle is the connectedness theorem: a set that is connected in the Zariski topology is also connected in the classical one. Results of this kind are explained in more detail in the article on the cohomology of algebraic varieties. They enable us to apply to complex algebraic varieties the methods of algebraic topology (homotopy, cohomology, etc.) and analysis (periods of integrals, Hodge theory); these methods are presented in Griffiths-Harris [1978]. Transcendental methods act as a powerful incentive to search for algebraic analogues and thus contribute to the subsequent development of abstract algebraic geometry.

**2.8. Localization.** The Zariski topology makes it possible to define regular functions in a more local fashion. Let $U \subset X$ be an open subset of an affine variety $X$, and $f \in K[X]$ a function that does not vanish at any point of $U$. Then the function $1/f$ is defined at every point of $U$ and can be considered a 'regular' function on $U$ in view of its algebraic origin (cf. Sect. 1.2). We must then also regard as regular the functions of the form $g/f$, where $g \in K[X]$.

More generally, we say that a function $h\colon U \to K$ is *regular at a point* $x \in U$ if there exist two functions $f, g \in K[X]$ such that $f(x) \neq 0$ and $h = g/f$ in some neighbourhood of $x$. More precisely we can say that $h$ coincides with $g/f$ on the set $U \cap \mathcal{D}(f)$, where $\mathcal{D}(f) = X - V(f) = \{x' \in X,\ f(x') \neq 0\}$. The sets of the form $\mathcal{D}(f)$ are called the *basic open subsets* of $X$. Clearly, they form a basis for the Zariski topology on $X$.

The functions on $U$ that are regular at every point of $U$ form a ring, which is denoted by $\mathcal{O}_X(U)$. If $U' \subset U$ then the restriction of functions from $U$ to $U'$ yields a homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(U')$ of rings (or of $K$-algebras). This object $\mathcal{O}_X$ – which will play an important role later on – is called the structure sheaf of rings on $X$. Clearly, $K[X] \subset \mathcal{O}_X(X)$; as a matter of fact, equality holds.

**Proposition.** *If $X$ is an affine variety then $K[X] = \mathcal{O}_X(X)$.*

Indeed, suppose the function $h\colon X \to K$ is regular at every point $x \in X$. Then $h = g_x/f_x$ in $\mathcal{D}(f_x)$ and $f_x(x) \neq 0$. By Hilbert's Nullstellensatz, the functions $f_x$, $x \in X$, generate the unit ideal in $K[X]$. Hence there exists a decomposition $1 = \Sigma\, a_x f_x$, with $a_x \in K[X]$. It follows that $h = h \cdot 1 = \Sigma\, a_x h f_x = \Sigma\, a_x g_x \in K[X]$.

This proposition allows us to talk about regular functions with no risk of ambiguity. The decomposition $1 = \Sigma\, a_x f_x$ plays a role similar to that of a partition of unity in the theory of differentiable manifolds.

**2.9. Quasi-affine Varieties.** Let again $U$ be an open subset of an affine variety $X$. In general the pair $(U, \mathcal{O}_X(U))$ is not an affine variety. First of all, the $K$-algebra $\mathcal{O}_X(U)$ may not be finitely generated. Secondly, there may be 'few' points in $U$, that is, the mapping $U \to \operatorname{Specm} \mathcal{O}_X(U)$ (see Sect. 2.2) may not be surjective.

**Example.** We want to show that $U = \mathbb{A}^n - \{0\}$ is not affine for $n \geq 2$. To this effect we shall verify that $\mathcal{O}_{\mathbb{A}^n}(U)$ coincides with $K[\mathbb{A}^n]$. In other words, every regular function on $\mathbb{A}^n - \{0\}$ extends to a regular function on $\mathbb{A}^n$. This property is reminiscent of Hartogs's theorem in the theory of analytic functions, and departs sharply from the situation that prevails in the differentiable case.

Indeed, let $f$ be a function regular on $U$. We cover $U$ by the sets $\mathcal{D}(T_i)$, the $T_i$ being coordinates on $\mathbb{A}^n$. Then the restriction of $f$ to $\mathcal{D}(T_i)$ is of the form $g_i/T_i^{r_i}$, with $g_i \in K[T_1, \ldots, T_n]$ and $r_i \geq 0$; we may further assume that $g_i$ is not divisible by $T_i$. Since the restrictions coincide on $\mathcal{D}(T_1) \cap \mathcal{D}(T_2)$, we see that $T_1^{r_1} g_2 = T_2^{r_2} g_1$. Now, from the uniqueness of the decomposition into prime factors in the polynomial ring $K[T_1, \ldots, T_n]$, we conclude that $r_1 = r_2 = 0$ and $g_1 = g_2 = f$.

On the other hand, the basic open sets $\mathcal{D}(f) \subset X$ are affine varieties. Related to this we have the following two facts: The ring $\mathcal{O}_X(\mathcal{D}(f))$ of regular functions on $\mathcal{D}(f)$ coincides with the ring $K[X][f^{-1}]$ of fractions of the form $g/f^r$, with $g \in K[X]$ and $r \geq 0$. Further, the Zariski topology on $\mathcal{D}(f)$ is induced by the Zariski topology on $X$.

In any case, the open subsets of affine varieties look locally like affine varieties. They are called *quasi-affine algebraic varieties.*

**2.10. Affine Algebraic Geometry.** Though algebraic geometry deals chiefly with projective varieties, it is worth mentioning that affine algebraic geometry also has its own, often unexpectedly hard, problems. Difficulties arise already for the simplest affine varieties, namely, affine space $\mathbb{A}^n$. Serre's problem on vector bundles over $\mathbb{A}^n$ was solved only comparatively recently (Suslin [1976], Quillen [1976]). Here is another famous question: suppose the variety $X \times \mathbb{A}^m$ is isomorphic to $\mathbb{A}^{n+m}$; is it true that $X$ is isomorphic to $\mathbb{A}^n$ ? An affirmative answer (which is obvious for $n = 1$) was obtained only recently for $n = 2$ (Miyanishi [1981]); for $n > 2$ the question is open.

Perhaps the reason for the difficulties lies in the fact that the space $\mathbb{A}^n$ (at least for $n > 1$) is very 'flexible'. The automorphisms of $\mathbb{A}^1$ are easily seen to be of the form $T' = aT + b$, with $a, b \in K$ and $a \neq 0$. That $\mathbb{A}^n$ (for $n > 1$) has quite a few more automorphisms is made clear by the example of the triangular transformation:

$$\begin{aligned} T_1' &= T_1 + f_0, \\ T_2' &= T_2 + f_1(T_1), \\ &\dots\dots\dots\dots\dots \\ T_n' &= T_n + f_{n-1}(T_1, \dots, T_{n-1}), \end{aligned}$$

where $f_i \in K[T_1, \dots, T_i]$. In particular, every finite subset of $\mathbb{A}^n$, where $n > 1$, can be carried by an automorphism into any other finite subset with the same cardinality. For $n = 2$ every automorphism of $\mathbb{A}^n$ is generated by triangular and linear automorphisms. This is not known, and almost certainly false, for $n > 2$. These questions are closely related to the problem of linearizing the action of algebraic groups on $\mathbb{A}^n$.

Finally, one should mention the so-called Jacobian problem. Consider a map $f\colon \mathbb{C}^n \to \mathbb{C}^n$ defined by polynomials $f_1, \dots, f_n$ in $\mathbb{C}[T_1, \dots, T_n]$, and suppose the Jacobian $\det(\partial f_j / \partial T_i)$ does not vanish anywhere on $\mathbb{C}^n$. (We may assume it is identically equal to 1.) The Jacobian conjecture says that $f$ must then be an isomorphism. For a discussion of this problem see Bass, Connell & Wright [1982].