§ 3. Algebraic Varieties

It was already perceived quite a long time ago that, by considering only affine varieties, one gets an incomplete picture of what goes on, as if one could see only part of the actual variety. This is connected with the fact that affine space is non-compact: we do not control the behaviour 'at infinity'. For instance, any two lines in the affine plane meet, unless they are parallel. It is convenient to postulate that even parallel lines meet, albeit in an 'infinitely distant point'. Adjoining these points to affine space $\mathbb{A}^n$ makes it into projective space $\mathbb{P}^n$. Another nice feature of the projective viewpoint is that affinely different curves, such as the ellipse, the parabola, and the hyperbola, turn out to be simply different affine parts of the projective conic. That is why algebraic geometry has always been preeminently a projective geometry. So we have to proceed now from affine varieties to the more general algebraic varieties.

3.1. Projective Space. The easiest way to define n-dimensional *projective space* $\mathbb{P}^n$ is to say it is the set of lines in the vector space K^{n+1}. Every line, that is, every one-dimensional vector subspace $L \subset K^{n+1}$, is given by a nonzero vector $(x_0, \ldots, x_n) \in K^{n+1}$, which is determined up to multiplication by a nonzero constant $\lambda \in K^* = K - \{0\}$. Therefore we may regard $\mathbb{P}^n$ as the quotient space $K^{n+1} - \{0\}/K^*$.

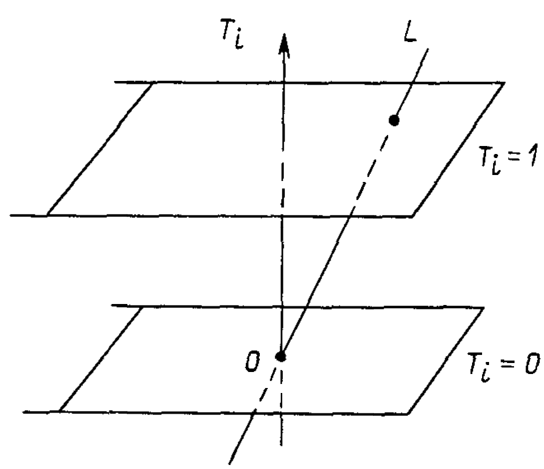

Fig. 5

The coordinate functions $T_0, \ldots, T_n$ on K^{n+1} are called the *homogeneous coordinates* on $\mathbb{P}^n$. However, one must be careful that the T_i, like any nonconstant polynomial in the T_i, are not functions on $\mathbb{P}^n$. Such expressions as T_j/T_i can be viewed as functions, but not on the whole of $\mathbb{P}^n$: only on the subset $U_i = \mathbb{P}^n - H_i$, where H_i consists of the points $(x_0, \ldots, x_n)$ with $x_i = 0$. In other words, U_i consists of those lines $L \subset K^{n+1}$ which project isomorphically onto the i-th coordinate axis. For fixed i, the functions $\xi_j^{(i)} = T_j T_i^{-1}$,

$j = 0, 1, \ldots, n$, define a one-to-one correspondence between U_i and the affine subspace $T_i = 1$ in K^{n+1}. Under this correspondence, H_i consists of the lines L lying in the hyperplane $T_i = 0$ and can be identified with $\mathbb{P}^{n-1}$. In this sense, $\mathbb{P}^n$ is obtained from the affine space $U_i \simeq K^n$ by adjunction of the hyperplane at infinity $H_i \simeq \mathbb{P}^{n-1}$.

The sets U_i form a covering of $\mathbb{P}^n$, and each of them has a natural structure of affine variety $\mathbb{A}^n$. Moreover, these structures agree on the intersections $U_i \cap U_j$. Indeed, we can regard $U_i \cap U_j$ as being the basic open set $\mathcal{D}(\xi_j^{(i)})$ in U_i, and also as the basic open subset $\mathcal{D}(\xi_i^{(j)})$ of U_j. In the former case, the ring of regular functions is generated by $\xi_0^{(i)}, \ldots, \xi_n^{(i)}, \xi_j^{(i)^{-1}}$; in the latter, by $\xi_0^{(j)}, \ldots, \xi_n^{(j)}, \xi_i^{(j)^{-1}}$. Now, these rings coincide. For example,

$$\xi_k^{(i)} = T_k/T_i = (T_k/T_j)(T_i/T_j)^{-1} = \xi_k^{(j)} \cdot \xi_i^{(j)^{-1}}$$

and

$$\xi_j^{(i)^{-1}} = (T_j/T_i)^{-1} = T_i/T_j = \xi_i^{(j)}.$$

Conversely, the $\xi^{(j)}$ can be expressed by means of the $\xi^{(i)}$.

Thus we see that $\mathbb{P}^n$ looks locally like an affine variety. We may therefore talk about regular functions on $\mathbb{P}^n$ (admittedly, they are quite scarce: constants only), or about the algebraic subvarieties of $\mathbb{P}^n$ (which are quite numerous), its Zariski topology, etc. Some comparable ideas can be used not only for $\mathbb{P}^n$, but also for any geometric object that looks locally like an affine variety. The resulting theory of algebraic varieties has much resemblance to that of differentiable or analytic manifolds.

3.2. Atlases and Varieties. Let X be a topological space. An *affine chart* (or *coordinate neighbourhood*) in X is an open subset $U \subset X$ equipped with a structure of affine variety, with the requirement that the induced topology on U should coincide with the Zariski topology. We say that two charts, U and U', are *compatible* if, for every open subset $V \subset U \cap U'$, one has $\mathcal{O}_U(V) = \mathcal{O}_{U'}(V)$.

An *atlas* on X is a collection $\mathcal{A} = (U_i)_{i \in I}$ of mutually compatible affine charts covering X. Two atlases, $\mathcal{A}$ and $\mathcal{A}'$, are *equivalent* if their union is also an atlas, that is, if the charts of $\mathcal{A}$ are compatible with those of $\mathcal{A}'$.

By a structure of *algebraic variety* on X we mean an equivalence class of atlases. In what follows we shall restrict attention to the algebraic varieties that have a *finite* atlas. By a chart on X we mean an affine chart that belongs to some atlas defining the variety structure of X. Every point lies in an arbitrarily small chart.

Every affine variety is an algebraic variety. Every closed subset $Y \subset X$ of an algebraic variety comes equipped with a canonical structure of algebraic variety; Y is also called a *subvariety* (or a closed subvariety) of X. An open subset $U \subset X$ also has an obvious structure of algebraic variety.

The covering of projective space $\mathbb{P}^n$ by the U_i is an atlas and converts $\mathbb{P}^n$ into an algebraic variety. More generally, if V is any finite-dimensional vector space over K, we denote by $\mathbb{P}(V)$ the set of lines of V through the origin. If $l\colon V \to K$ is a nonzero linear map, we define $H_l \subset \mathbb{P}(V)$ to be the set of lines $L \subset \ker l$. Then $U_l = \mathbb{P}(V) - H_l$ consists of those lines L for which $l(L) = K$ and can be identified with the affine subspace $l^{-1}(1) \subset V$. The structures on the various U_l are compatible and make $\mathbb{P}(V)$ into an algebraic variety. Of course $\mathbb{P}^n = \mathbb{P}(K^{n+1})$.

3.3. Gluing. This operation yields some new varieties out of old ones. Let (X_i) be a finite covering of some set X, where each X_i has a structure of algebraic variety. We make two assumptions:

a) for every pair i, j the set $X_i \cap X_j$ is open in X_i and in X_j;
b) the algebraic variety structures induced on $X_i \cap X_j$ from X_i and from X_j coincide.

Then there exists on X a unique structure of algebraic variety such that the X_i are open subvarieties. We say that X is obtained by *gluing* (or pasting together) the varieties X_i.

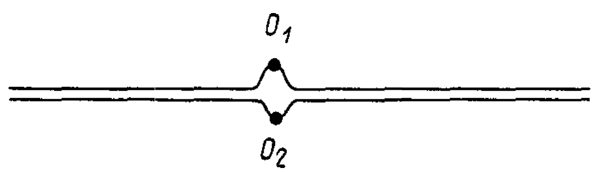

Fig. 6

One may, for instance, think of projective space $\mathbb{P}^n$ as the result of gluing the affine spaces U_i, $\quad i = 0, 1, \dots, n$. Here is another example. Suppose X_1 and X_2 are isomorphic to the affine line $\mathbb{A}^1$, and let T_1 and T_2 be coordinates on X_1, respectively, X_2. Let us identify $X_1 - \{0\}$ and $X_2 - \{0\}$ by setting $T_1 = T_2$. What we get is an *affine line with the point* 0 *doubled* (see Fig. 6). Such a variety occurs naturally as the set of orbits for the action $\lambda(x, y) = (\lambda x, \lambda^{-1} y)$ of the group K^* on the plane K^2.

Example. A good exercise on the theme of gluing is the construction of *torus embeddings*. We fix a lattice M, that is, a free abelian group of finite type (which is therefore isomorphic to $\mathbb{Z}^n$, but the basis is irrelevant). Let $S \subset M$ be a submonoid, that is, S contains 0 and is closed under addition. Then we can form the semigroup K-algebra $K[S]$. It is generated additively by all elements of the form x^m, with $m \in S$, multiplication being defined by the rule $x^m \cdot x^{m'} = x^{m+m'}$. If S is finitely generated as a monoid then the K-algebra $K[S]$ is of finite type and defines an affine variety, namely, $\operatorname{Specm} K[S]$. For instance, if $S = M$ we get the n-dimensional torus $\mathbb{T} = \operatorname{Specm} K[M] = \operatorname{Specm}[T_1, \dots, T_n, T_1^{-1}, \dots, T_n^{-1}]$.

We consider now in the dual lattice $M^* = \mathrm{Hom}(M, \mathbb{Z})$ a subset B which is contained in a $\mathbb{Z}$-basis of the group M^*. We can attach to it the following monoid in M:

$$B^\perp = \{m \in M, \quad b(m) \geq 0 \quad \forall b \in B\}.$$

The corresponding affine variety $\mathrm{Specm}\, K[B^\perp]$ will be denoted by X_B. (This variety is called a torus embedding because the torus $\mathbb{T}$ acts on it in a natural way.) If $B' \subset B$ then $B'^\perp \supset B^\perp$, which gives rise to a natural homomorphism of K-algebras $K[B^\perp] \to K[B'^\perp]$ and to the opposite morphism of varieties $X_{B'} \to X_B$. It is not difficult to check that the latter is an open immersion.

Now, given a collection Σ of such subbases B of M^*, it is possible to glue together the varieties X_B and $X_{B'}$ $(B, B' \in \Sigma)$ along the open pieces $X_{B \cap B'}$, so as to obtain a torus embedding X_Σ. For instance, $\mathbb{P}^n$ is obtained from $\Sigma = \{B_0, \dots, B_n\}$, where $B_0 = \{e_1, \dots, e_n\}$, and

$$B_i = \{e_1, \dots, \widehat{e}_i, \dots, e_n, -e_1 - \dots - e_n\} \quad \text{for } i = 1, \dots, n.$$

What makes the interest of torus embeddings, is that various objects on X_Σ (like invertible sheaves and their cohomology, differential forms, etc.) can be described in combinatorial terms depending on Σ. For instance, invertible sheaves are represented by polyhedra in $M \otimes \mathbb{R}$, and their sections by the integer points on these polyhedra. For further details, see Danilov [1978].

3.4. The Grassmann Variety. Let again V be a vector space over K. We denote by $G(k, V)$ (or $G(k, n)$ if $n = \dim V$) the set of k-dimensional subspaces $W \subset V$; for $k = 1$ we get $\mathbb{P}(V)$. Generalizing the construction of projective space, we shall give $G(k, V)$ the structure of an algebraic variety, called the *Grassmann variety.*

Let $V = V' \oplus V''$ be a direct decomposition, with $\dim V' = k$. To each such decomposition we shall attach the set $U(V', V'')$, consisting of the subspaces $W \subset V$ which project isomorphically onto V'. These subspaces can be identified with the graphs of linear maps from V' to V''. Hence $U(V', V'') \simeq \mathrm{Hom}_k(V', V'') \simeq V'' \otimes V'^*$ is naturally identified with a vector space of dimension $k(n-k)$ and is endowed with the structure of an affine variety. It is an immediate verification that all these charts $U(V', V'')$ are compatible and give $G(k, V)$ an algebraic variety structure. For further details on the Grassmannian, see Griffiths-Harris [1978] and Grothendieck-Dieudonné [1971].

3.5. Projective Varieties. A closed subset of projective space is said to be a *projective variety.* We exhibit a general method for producing such varieties.

Let V be a vector space over K. We define a *cone* in V to be an algebraic subvariety $C \subset V$ which is invariant under scalar multiplication, that is, multiplication by a constant. To every cone C we associate the subset $\mathbb{P}(C) \subset \mathbb{P}(V)$ consisting of the lines $L \subset C$. The set $\mathbb{P}(C)$ is closed in $\mathbb{P}(V)$. Indeed, provided we identify a chart U_l (where $l \colon V \to K$ is a linear map) with the affine subspace $l^{-1}(1) \subset V$, the set $\mathbb{P}(C) \subset U_l$ is seen to be identical with the intersection $C \cap l^{-1}(1)$, which is obviously closed in $l^{-1}(1)$.

In the coordinates $T_0, \ldots, T_n$ on V, the cone C is given by homogeneous equations $f_j(T_0, \ldots, T_n) = 0, \quad j \in J$. Then $\mathbb{P}(C) \cap U_i$ is given by the equations $f_j(T_0/T_i, \ldots, T_n/T_i) = 0$. The equations $f_j = 0$ are called the *homogeneous equations* of $\mathbb{P}(C)$.

Conversely, every projective variety $X \subset \mathbb{P}(V)$ is of the form $\mathbb{P}(C)$ for some cone $C \subset V$. Indeed, let (U_i) be the standard atlas of $\mathbb{P}^n$, and suppose $X \cap U_i$ is given by equations $f_j^{(i)}(T_0/T_i, \ldots, T_n/T_i) = 0, \quad j \in J_i$. Then, for large m, $T_i^m f_j^{(i)}(T_0/T_i, \ldots, T_n/T_i) = g_j^{(i)}(T_0, \ldots, T_n)$ is a homogeneous form in $T_0, \ldots, T_n$, and the equations $g_j^{(i)} = 0$, $j \in J_i$, $i = 0, 1, \ldots, n$, define X in $\mathbb{P}^n$.

The simplest projective varieties are the linear ones. If $W \subset V$ is a vector subspace, the subvariety $\mathbb{P}(W) \subset \mathbb{P}(V)$ is said to be *linear*. If W is a hyperplane in V then $\mathbb{P}(W)$ is called a *hyperplane* in $\mathbb{P}(V)$. We define the *linear hull* of a set to be the intersection of all the linear varieties that contain it. For two distinct points, x and y, it is nothing but the projective line $\overline{xy}$, and so forth. To give a hyperplane $W \subset V$ is the same as giving a line $W^\perp$ in the dual space V^*, and conversely. Hence the set of all hyperplanes in $\mathbb{P}(V)$ is also a projective space, namely, $\mathbb{P}(V^*)$.

Every vector space V can be regarded as an affine part of projective space $\mathbb{P}(V \oplus K)$, more precisely as the complementary set to the hyperplane $\mathbb{P}(V) \subset \mathbb{P}(V \oplus K)$. If $X \subset V$ is an algebraic variety then the closure of X in $\mathbb{P}(V \oplus K)$ is a projective variety. This is a standard way to proceed from affine to projective varieties (which, by the way, depends on the embedding $X \subset V$). If $\xi_1, \ldots, \xi_n$ are coordinates on V, projectivization looks as follows. Let $f(\xi_1, \ldots, \xi_n)$ be a polynomial of degree d; its homogenization is the homogeneous, degree d polynomial $\tilde{f}(T_0, \ldots, T_n) = T_0^d f(T_1/T_0, \ldots, T_n/T_0)$. Now if X is given by equations $f_j = 0$ then its projectivization $\overline{X}$ is defined by the equations $\tilde{f}_j = 0$.

§ 4. Morphisms of Algebraic Varieties

4.1. Definitions. Let X be an algebraic variety described by an atlas (X_i), and Y an affine variety. We say that a map $f\colon X \to Y$ is *regular* if the restriction of f to every chart X_i has this property. In particular, we have the notion of a regular function. For any open set $U \subset X$, we denote by $\mathcal{O}_X(U)$ the K-algebra of functions regular on U. If $U' \subset U$, there is a restriction homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(U')$.

Suppose now Y is an arbitrary algebraic variety. A continuous mapping $f\colon X \to Y$ is called a *morphism* (or a regular map) *of algebraic varieties* if, for every chart $V \subset Y$, the induced mapping $f^{-1}(V) \to V$ is regular. In other words, for every regular function g on the open subset $V \subset Y$, the function $f^*(g) = g \circ f$ must be regular on $f^{-1}(V)$. This means that f^* yields an algebra homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$.

The composite of two morphisms is again a morphism, so that algebraic varieties form a category. The canonical injection of a closed subvariety is a morphism, and we say that a morphism $Y \to X$ is a *closed immersion* if it yields an isomorphism of Y onto a closed subvariety of X. If $f\colon X \to Y$ is a morphism, and $Y' \subset Y$ a closed subvariety, then $f^{-1}(Y')$ is a closed subvariety of X (cf. Sect. 2.4). In particular, for a point $y \in Y$ the variety $f^{-1}(y) \subset X$ is called the *fibre of the morphism* f over y.

A variety X provided with a morphism $f\colon X \to Y$ is sometimes called a variety over Y, or a Y-variety. X is thereby viewed as the family of algebraic varieties $X_y = f^{-1}(y)$, parametrized by the points $y \in Y$. Given two Y-varieties, say, $f\colon X \to Y$ and $f'\colon X' \to Y$, a morphism from f to f' is a morphism $\varphi\colon X \to X'$ such that $f = f' \circ \varphi$. Each fibre $f^{-1}(y)$ is mapped into the corresponding fibre $f'^{-1}(y)$, so we get a family of morphisms $\varphi_y\colon X_y \to X'_y$.

4.2. Products of Varieties. Let X and Y be two algebraic varieties, with defining atlases (X_i) and (Y_j). Then $(X_i \times Y_j)$ is an atlas for the product $X \times Y$, so $X \times Y$ is also an algebraic variety. An easy verification shows that $X \times Y$ is the direct product of X and Y in the category of varieties.

In particular, for any variety X, the diagonal mapping $\Delta\colon X \to X \times X$ $(\Delta(x) = (x, x))$ is a morphism, though in general it is not a closed immersion. In other words, the diagonal in $X \times X$ may fail to be closed. An example is furnished by the 'affine line with a point doubled' from Sect. 3.3. If, in spite of that, the diagonal in $X \times X$ is closed then we say that the variety X is *separated.* (One should not confuse this notion with the question whether X is Hausdorff as a topological space!) Any affine variety, for instance, is separated (cf. Sect. 2.4). The class of separated varieties is closed under taking direct products or going over to subvarieties. We will check below that projective space – and hence any projective variety – is separated. In what follows we shall therefore deal exclusively with separated varieties.

That a variety is non-separated has to do with the fact that, when we obtain it by gluing some of its affine pieces, these are glued imperfectly. To be precise, one has the following *separatedness criterion:* a variety X, described by an atlas (X_i), is separated if and only if the image of $X_i \cap X_j$ under the canonical injection into $X_i \times X_j$ is closed. In fact, the image of $X_i \cap X_j$ in $X_i \times X_j$ is just the intersection of $X_i \times X_j$ with the diagonal in $X \times X$.

Let us apply this criterion to the standard atlas (U_i), $i = 0, 1, \ldots, n$, of projective space $\mathbb{P}^n$ (cf. Sect 3.1). It is easy to check that the image of $U_i \cap U_j$ in $U_i \times U_j$ is given by the equations

$$\xi_k^{(i)} = \xi_j^{(i)} \xi_k^{(j)}, \quad \xi_k^{(j)} = \xi_k^{(i)} \xi_i^{(j)}, \quad k = 0, \ldots, n,$$

whence we see that $\mathbb{P}^n$ is separated.