which is the same as saying that the homogeneous coordinate rings $S(X)$, $S(X')$ are isomorphic as graded K-algebras; while we say that they are isomorphic under the weaker condition that there is a biregular map between them. (We will see an explicit example where the two notions do not agree in Exercise 2.10.)

Exercise 2.3. Using the result of Exercise 2.2, show that for $n \geq 2$ the complement of the origin in $\mathbb{A}^n$ is not isomorphic to an affine variety.

Example 2.4. The Veronese Map

The construction of the rational normal curve can be further generalized: for any n and d, we define the *Veronese map* of degree d

$$v_d\colon \mathbb{P}^n \to \mathbb{P}^N$$

by sending

$$[X_0, \dots, X_n] \mapsto [\dots X^I \dots],$$

where X^I ranges over all monomials of degree d in $X_0, \dots, X_n$. As in the case of the rational normal curves, we will call the Veronese map any map differing from this by an automorphism of $\mathbb{P}^N$. Geometrically, the Veronese map is characterized by the property that the hypersurfaces of degree d in $\mathbb{P}^n$ are exactly the hyperplane sections of the image $v_d(\mathbb{P}^n) \subset \mathbb{P}^N$. It is not hard to see that the image of the Veronese map is an algebraic variety, often called a *Veronese variety*.

Exercise 2.5. Show that the number of monomials of degree d in $n + 1$ variables is the binomial coefficient $\binom{n+d}{d}$, so that the integer N is $\binom{n+d}{d} - 1$.

For example, in the simplest case other than the case $n = 1$ of the rational normal curve, the quadratic Veronese map

$$v_2\colon \mathbb{P}^2 \to \mathbb{P}^5$$

is given by

$$v^2\colon [X_0, X_1, X_2] \mapsto [X_0^2, X_1^2, X_2^2, X_0X_1, X_0X_2, X_1X_2].$$

The image of this map, often called simply the *Veronese surface*, is one variety we will encounter often in the course of this book.

The Veronese variety $v_d(\mathbb{P}^n)$ lies on a number of obvious quadric hypersurfaces: for every quadruple of multi-indices I, J, K, and L such that the corresponding monomials $X^IX^J = K^KX^L$, we have a quadratic relation on the image. In fact, it is not hard to check that the Veronese variety is exactly the zero locus of these quadratic polynomials.

Example 2.6. Determinantal Representation of Veronese Varieties

The Veronese surface, that is, the image of the map $\nu^2\colon \mathbb{P}^2 \to \mathbb{P}^5$, can also be described as the locus of points $[Z_0, \ldots, Z_5] \in \mathbb{P}^5$ such that the matrix

$$\begin{bmatrix} Z_0 & Z_3 & Z_4 \\ Z_3 & Z_1 & Z_5 \\ Z_4 & Z_5 & Z_2 \end{bmatrix}$$

has rank 1. In general, if we let $\{Z_{i,j}\}_{0 \le i \le j \le n}$ be the coordinates on the target space of the quadratic Veronese map

$$\nu_2\colon \mathbb{P}^n \to \mathbb{P}^{(n+1)(n+2)/2-1},$$

then we can represent the image of ν_2 as the locus of the 2×2 minors of the $(n+1) \times (n+1)$ symmetric matrix with (i, j)th entry $Z_{i-1,j-1}$ for $i \le j$.

Example 2.7. Subvarieties of Veronese Varieties

The Veronese map may be applied not only to a projective space $\mathbb{P}^n$, but to any variety $X \subset \mathbb{P}^n$ by restriction. Observe in particular that if we restrict ν_d to a linear subspace $\Lambda \cong \mathbb{P}^k \subset \mathbb{P}^n$, we get just the Veronese map of degree d on $\mathbb{P}^k$. For example, the images under the map $\nu_2\colon \mathbb{P}^2 \to \mathbb{P}^5$ of lines in $\mathbb{P}^2$ give a family of conic plane curves on the Veronese surface S, with one such conic passing through any two points of S.

More generally, we claim that the image of a variety $Y \subset \mathbb{P}^n$ under the Veronese map is a subvariety of $\mathbb{P}^N$. To see this, note first that homogeneous polynomials of degree k in the homogeneous coordinates Z on $\mathbb{P}^N$ pull back to give (all) polynomials of degree $d \cdot k$ in the variables X. Next, observe (as in the remark following Exercise 1.3) that the zero locus of a polynomial $F(X)$ of degree m is also the common zero locus of the polynomials $\{X_i F(X)\}$ of degree $m + 1$. Thus a variety $Y \subset \mathbb{P}^n$ expressible as the common zero locus of polynomials of degree m and less may also be realized as the common zero locus of polynomials of degree exactly $k \cdot d$ for some k. It follows that its image $\nu_d(Y) \subset \mathbb{P}^N$ under the Veronese map is the intersection of the Veronese variety $\nu_d(\mathbb{P}^n)$—which we have already seen is a variety—with the common zero locus of polynomials of degree k.

For example, if $Y \subset \mathbb{P}^2$ is the curve given by the cubic polynomial $X_0^3 + X_1^3 + X_2^3$, then we can also write Y as the common locus of the quartics

$$X_0^4 + X_0 X_1^3 + X_0 X_2^3, \quad X_0^3 X_1 + X_1^4 + X_1 X_2^3, \quad \text{and} \quad X_0^3 X_2 + X_1^3 X_2 + X_2^4.$$

The image $\nu_2(Y) \subset \mathbb{P}^5$ is thus the intersection of the Veronese surface with the three quadric hypersurfaces

$$Z_0^2 + Z_1 Z_3 + Z_2 Z_4, \quad Z_0 Z_3 + Z_1^2 + Z_2 Z_5, \quad \text{and } Z_0 Z_4 + Z_1 Z_5 + Z_2^2.$$

In particular, it is the intersection of nine quadrics.

Exercise 2.8. Let $X \subset \mathbb{P}^n$ be a projective variety and $Y = v_d(X) \subset \mathbb{P}^N$ its image under the Veronese map. Show that X and Y are isomorphic, i.e., show that the inverse map is regular.

Exercise 2.9. Use the preceding analysis and exercise to deduce that any projective variety is isomorphic to an intersection of a Veronese variety with a linear space (and hence in particular that any projective variety is isomorphic to an intersection of quadrics).

Exercise 2.10. Let $X \subset \mathbb{P}^n$ be a projective variety and $Y = v_d(X) \subset \mathbb{P}^N$ its image under the Veronese map. What is the relation between the homogeneous coordinate rings of X and Y?

In case the field K has characteristic zero, Veronese map has a coordinate-free description that is worth bearing in mind. Briefly, if we view $\mathbb{P}^n = \mathbb{P}V$ as the space lines in a vector space V, then the Veronese map may be defined as the map

$$v_d\colon \mathbb{P}V \to \mathbb{P}(\mathrm{Sym}^d V)$$

to the projectivization of the dth symmetric power of V, given by

$$v_d\colon [v] \mapsto [v^d].$$

Equivalently, if we apply this to V^* rather than V, the image of the Veronese map may be viewed as the (projectivization of the) subset of the space $\mathrm{Sym}^d V^*$ of all polynomials on V consisting of dth powers of linear forms. Note that this is false for fields K of arbitrary characteristic: for example, if $\mathrm{char}(K) = p$, the locus in $\mathbb{P}(\mathrm{Sym}^p V)$ of pth powers of elements of V is not a rational normal curve, but a line. What is true in arbitrary characteristic is that the Veronese map v_d may be viewed as the map $\mathbb{P}V \to \mathbb{P}(\mathrm{Sym}^d V)$ sending a vector v to the linear functional on $\mathrm{Sym}^d V^*$ given by evaluation of polynomials at p.

Example 2.11. The Segre Maps

Another fundamental family of maps are the *Segre maps*

$$\sigma\colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$$

defined by sending a pair $([X], [Y])$ to the point in $\mathbb{P}^{(n+1)(m+1)-1}$ whose coordinates are the pairwise products of the coordinates of $[X]$ and $[Y]$, i.e.,

$$\sigma\colon ([X_0, \ldots, X_n], [Y_0, \ldots, Y_m]) \mapsto [\ldots, X_i Y_j, \ldots],$$

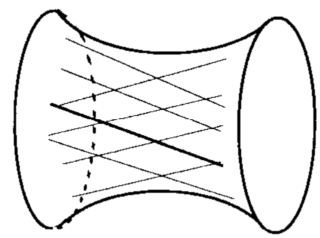
where the coordinates in the target space range over all pairwise products of coordinates X_i and Y_j.

It is not hard to see that the image of the Segre map is an algebraic variety, called a *Segre variety*, and sometimes denoted $\Sigma_{n,m}$: if we label the coordinates on the target space as $Z_{i,j}$, we see that it is the common zero locus of the quadratic polynomials $Z_{i,j} \cdot Z_{k,l} - Z_{i,l} \cdot Z_{k,j}$. (In particular, the Segre variety is another example of a *determinantal variety*; it is the zero locus of the 2×2 minors of the matrix $(Z_{i,j})$.)

The first example of a Segre variety is the variety $\Sigma_{1,1} = \sigma(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^3$, that is, the image of the map

$$\sigma: ([X_0, X_1], [Y_0, Y_1]) \mapsto [X_0 Y_0, X_0 Y_1, X_1 Y_0, X_1 Y_1].$$

This is the locus of the single quadratic polynomial $Z_0 Z_3 - Z_1 Z_2$, that is, it is simply a quadric surface. Note that the fibers of the two projection maps from $\mathbb{P}^n \times \mathbb{P}^m$ to $\mathbb{P}^n$ and $\mathbb{P}^m$ are carried, under σ, into linear subspaces of $\mathbb{P}^{(n+1)(m+1)-1}$; in particular, the fibers of $\mathbb{P}^1 \times \mathbb{P}^1$ are carried into the families of lines $\{Z_1 = \lambda Z_0, Z_3 = \lambda Z_2\}$ and $\{Z_2 = \lambda Z_0, Z_3 = \lambda Z_1\}$. Note also that the description of the polynomial $Z_0 Z_3 - Z_1 Z_2$ as the determinant of the matrix



$$M = \begin{pmatrix} Z_0 & Z_1 \\ Z_2 & Z_3 \end{pmatrix}$$

displays the two families of lines nicely: one family consists of lines where the two columns satisfy a given linear relation, the other lines where the two rows satisfy a given linear relation.

Another common example of a Segre variety is the image

$$\Sigma_{2,1} = \sigma(\mathbb{P}^2 \times \mathbb{P}^1) \subset \mathbb{P}^5,$$

called the *Segre threefold*. We will encounter it again several times (for example, it is an example of a *rational normal scroll*, and as such is denoted $X_{1,1,1}$). For now, we mention the following facts.

Exercise 2.12. (i) Let L, M, and $N \subset \mathbb{P}^3$ be any three pairwise skew (i.e., disjoint) lines. Show that the union of the lines in $\mathbb{P}^3$ meeting all three lines is projectively equivalent to the Segre variety $\Sigma_{1,1} \subset \mathbb{P}^3$ and that this union is the unique Segre variety containing L, M, and N. (*)

(ii) More generally, suppose that L, M, and N are any three pairwise disjoint $(k-1)$-planes in $\mathbb{P}^{2k-1}$. Show that the union of all lines meeting L, M, and N is projectively equivalent to the Segre variety $\Sigma_{k-1,1} \subset \mathbb{P}^{2k-1}$ and that this union is the unique Segre variety containing L, M, and N. Is there an analogous description of Segre varieties $\Sigma_{a,b}$ with $a, b \geq 2$?

Exercise 2.13. Show that the twisted cubic curve $C \subset \mathbb{P}^3$ may be realized as the intersection of the Segre threefold with a three-plane $\mathbb{P}^3 \subset \mathbb{P}^5$.

Exercise 2.14. Show that any line $l \subset \Sigma_{2,1} \subset \mathbb{P}^5$ is contained in the image of a fiber of $\mathbb{P}^2 \times \mathbb{P}^1$ over $\mathbb{P}^2$ or $\mathbb{P}^1$ (*). (The same is true for any linear space contained in any Segre variety $\sigma(\mathbb{P}^n \times \mathbb{P}^m)$, but we will defer the most general statement until Theorem 9.22.)

Exercise 2.15. Show that the image of the diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$ under the Segre map is the Veronese variety $v_2(\mathbb{P}^n)$, lying in a subspace of $\mathbb{P}^{n^2+2n}$; deduce that in general the diagonal $\Delta_X \subset X \times X$ in the product of any variety with itself is a subvariety of that product, and likewise for all diagonals in the n fold product X^n.

Example 2.16. Subvarieties of Segre Varieties

Having given the product $\mathbb{P}^n \times \mathbb{P}^m$ the structure of a projective variety, a natural question to ask is how we may describe its subvarieties. A naive answer is immediate. To begin with, we say that a polynomial $F(Z_0, \ldots, Z_n, W_0, \ldots, W_m)$ in two sets of variables is *bihomogeneous of bidegree* (d, e) if it is simultaneously homogeneous of degree d in the first set of variables and of degree e in the second, that is, of the form

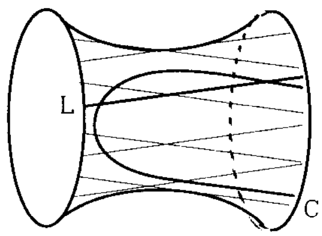
$$F(Z, W) = \sum_{\substack{I,J:\\ \Sigma i_\alpha = d,\\ \Sigma j_\beta = e}} a_{I,J} \cdot Z_0^{i_0} \ldots Z_n^{i_n} \cdot W_0^{j_0} \ldots W_m^{j_m}.$$

Now, since polynomials of degree d on the target projective space $\mathbb{P}^{(m+1)(n+1)-1}$ pull back to polynomials $F(Z, W)$ that are bihomogeneous of bidegree (d, d), the obvious answer is that subvarieties of $\mathbb{P}^n \times \mathbb{P}^m$ are simply the common zero loci of such polynomials (observe that the zero locus of any bihomogeneous polynomial is a well-defined subset of $\mathbb{P}^n \times \mathbb{P}^m$). At the same time, as in the discussion of subvarieties of the Veronese variety, we can see that the zero locus of a bihomogeneous polynomial $F(Z, W)$ of bidegree (d, e) is the common zero locus of the bihomogeneous polynomials of degree (d', e') divisible by it, for any $d' \geq d$ and $e' \geq e$; so that more generally we can say that the subvarieties of a Segre variety $\mathbb{P}^n \times \mathbb{P}^m$ are the zero loci of bihomogeneous polynomials of any bidegrees.

As an example, consider the twisted cubic $C \subset \mathbb{P}^3$ of Example 1.10 given as the image of the map

$$t \mapsto [1, t, t^2, t^3].$$

As we observed before, C lies on the quadric surface $Z_0 Z_3 - Z_1 Z_2 = 0$, which we now recognize as the Segre surface $S = \sigma(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^3$. Now, restrict to S the other two quadratic polynomials defining the twisted cubic. To begin with, the polynomial $Z_0 Z_2 - Z_1^2$ on $\mathbb{P}^3$ pulls back to $X_0 X_1 Y_0^2 - X_0^2 Y_1^2$, which factors into a product of X_0 and $F(X, Y) = X_1 Y_0^2 - X_0 Y_1^2$. The zero locus of this polynomial is thus the union of the twisted cubic with the line on S given by $X_0 = 0$ (or equivalently by $Z_0 = Z_1 = 0$). On the other hand, the polynomial $Z_1 Z_3 - Z_2^2$ pulls back to $X_0 X_1 Y_1^2 - X_1^2 Y_0^2$, which factors as $-X_1 \cdot F$; so its zero locus is the union of the curve C and the line $Z_2 = Z_3 = 0$. In sum, then, the twisted cubic curve is the zero locus of a single bihomogeneous polynomial $F(X, Y)$ of bidegree (1, 2) on the Segre surface $S = \sigma(\mathbb{P}^1 \times \mathbb{P}^1)$; the quadratic polynomials defining C restrict to S to give the bihomogeneous polynomials of bidegree (2, 2) divisible by F (equivalently, the quadric surfaces containing C cut on S the unions of C with the lines of one ruling of S).



Exercise 2.17. Conversely, let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the zero locus of an irreducible bihomogeneous polynomial $F(X, Y)$ of bidegree (1, 2). Show that the image of C under the Segre map

$$\sigma: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$$

is a twisted cubic curve.

Exercise 2.18. Now let $C = C_{\alpha,\beta} \subset \mathbb{P}^3$ be a rational quartic curve, as introduced in Example 1.26. Observe that C lies on the Segre surface S given by $Z_0Z_3 - Z_1Z_2 = 0$, that S is the unique quadric surface containing C, and that C is the zero locus of a bihomogeneous polynomial of bidegree (1, 3) on $S \cong \mathbb{P}^1 \times \mathbb{P}^1$. Use this to do Exercise 1.29.

Exercise 2.19. Use the preceding exercise to show in particular that there is a continuous family of curves $C_{\alpha,\beta}$ not projectively equivalent to one another. (*)

Exercise 2.20. a. Let $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ be projective varieties. Show that the image $\sigma(X \times Y) \subset \sigma(\mathbb{P}^n \times \mathbb{P}^m) \subset \mathbb{P}^{nm+n+m}$ of the Segre map restricted to $X \times Y$ is a projective variety. b. Now suppose only that $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ are quasi-projective. Show that $\sigma(X \times Y)$ is likewise quasi-projective, that is, locally closed in $\mathbb{P}^{nm+n+m}$.

Example 2.21. Products of Varieties

At the outset of Example 2.11, we referred to the product $\mathbb{P}^n \times \mathbb{P}^m$; we can only mean the product as a set. This space does not a priori have the structure of an algebraic variety. The Segre embedding, however, gives it one, which we will adopt as a definition of the product as a variety. In other words, when we talk about "the variety $\mathbb{P}^n \times \mathbb{P}^m$" we mean the image of the Segre map. Similarly, if $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ are locally closed, according to Exercise 2.20, the image of the product $X \times Y \subset \mathbb{P}^n \times \mathbb{P}^m$ is a locally closed subset of $\mathbb{P}^{nm+n+m}$, which we will take as the definition of "the product $X \times Y$" as a variety.

A key point to be made in connection with this definition is that this is actually a *categorical product*, i.e., the projection maps $\pi_X: X \times Y \to X$ and $\pi_Y: X \times Y \to Y$ are regular and the variety $X \times Y$, together with these projection maps, satisfies the conditions for a product in the category of quasi-projective varieties and regular