# ALGEBRAIC GEOMETRY: HOMEWORK 4

*This homework is due by 5pm on August 23.*

(1) (Affine charts) Let $Q = V(XY - ZW) \subset \mathbb{P}^3$. Write the four affine charts for $Q$ and the transition functions between *one* pair of them.

(2) (Projective closure) Think of $\mathbb{A}^n$ as the open subset of $\mathbb{P}^n$ where the last homogeneous coordinate is non-zero. Find (with proof!) the closure in $\mathbb{P}^n$ of the following varieties in $\mathbb{A}^n$, and identify the points at infinity in the closure.
  (a) $V(x^2 + y^2 - 1) \subset \mathbb{A}^2$,
  (b) $V(y - x^2, z - x^3) \subset \mathbb{A}^3$.

  (The general statement is as follows. The closure $\overline{X}$ is given by
  $$\overline{X} = V(\{p^{\text{hom}} \mid p \in I(X)\}),$$
  where $p^{\text{hom}}$ is the homogenization of $p$ with respect to last coordinate variable. You should be able to prove this, but you do not have to for this homework.)

(3) (5 points define a conic) Let $X \subset \mathbb{P}^2$ be a set of 5 points, no 3 on a line. Prove that there is a unique conic containing $X$.

(4) (Pencil of conics) Let $F$ and $G$ be irreducible homogeneous degree 2 polynomials in $k[X, Y, Z]$. For each $[s : t] \in \mathbb{P}^1$, we get a conic $Q_{s:t} = V(sF + tG)$ (Such a family of conics is called a "pencil"). Suppose $V(F)$ and $V(G)$ intersect in 4 distinct points. Prove that exactly three members of the pencil are degenerate, and describe them in terms of the 4 points of intersection of $V(F)$ and $V(G)$.

**Glossary of terms in projective geometry.**

(1) A *line* in $\mathbb{P}^2$ is the vanishing set of a homogeneous linear polynomial. Easy linear algebra shows that any two distinct lines in $\mathbb{P}^2$ intersect in a unique point, and any two distinct points in $\mathbb{P}^2$ lie on a unique line. In $\mathbb{P}^3$, the vanishing set of homogeneous linear polynomial is called a *plane*; in higher dimensions, it is called a *hyperplane*.

(2) A *conic* in $\mathbb{P}^2$ is the vanishing set of a homogeneous quadratic polynomial. A conic is *non-degenerate* if the defining polynomial is irreducible, and *degenerate* otherwise. A degenerate conic is a union of two lines.

(3) When we think of $\mathbb{A}^n \subset \mathbb{P}^n$ as the subset where the last coordinate is non-zero (say), the complement is called the *hyperplane at infinity* and its points are called the *points at infinity*.