ALGEBRAIC GEOMETRY: HOMEWORK 5

(1) In this problem, consider $\mathbb{A}^k$ as the open subset of $\mathbb{P}^k$ where the last homogenous coordinate is non-zero.

The following maps from an open subset of $\mathbb{A}^n$ to $\mathbb{A}^m$ extend to regular maps from $\mathbb{P}^n$ to $\mathbb{P}^m$. Write down these extensions using homogeneous polynomials.

(a) $f\colon \mathbb{A}^1 \to \mathbb{A}^2$ defined by $f(t) = (t^2 - 1, t^3 - t)$.

(b) $f\colon \mathbb{A}^2 \setminus V(xy) \to \mathbb{A}^3$ defined by $f(x, y) = (x/y, y/x, 1/xy)$.

(2) Show that the natural map

$$\pi\colon \mathbb{A}^2 \setminus \{(0,0)\} \to \mathbb{P}^1$$

defined by $\pi(x, y) = [x : y]$ does not extend to a regular map $\pi\colon \mathbb{A}^2 \to \mathbb{P}^1$.

(3) (3-transitivity of PGL_2) Given three distinct points $p_1, p_2, p_3 \in \mathbb{P}^1$, prove that there exists a unique projective linear transformation $\mathbb{P}^1 \to \mathbb{P}^1$ that sends

$$0 = [0 : 1] \mapsto p_1, 1 = [1 : 1] \mapsto p_2, \text{ and } \infty = [1 : 0] \mapsto p_3.$$

(4) (A cubic surface as a conic fibration) Suppose $\operatorname{char} k \neq 2, 3$.

Let $S \subset \mathbb{P}^3$ be the Fermat cubic surface

$$S = V(X^3 + Y^3 + Z^3 + W^3).$$

(a) Consider the linear projection $\pi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ defined by

$$[X : Y : Z : W] \mapsto [X + Y, Z + W].$$

Show that the center L of the linear projection is contained in S.

(b) Show that $\pi\colon S \setminus L \to \mathbb{P}^1$ extends to a regular map $\pi\colon S \to \mathbb{P}^1$.

(c) What is the fiber of $\pi\colon S \to \mathbb{P}^1$ over a point $[a : b] \in \mathbb{P}^1$? (Be careful!)

(d) (Not to be turned in but highly recommended) Draw a (real) picture depicting L, S, a typical fiber of the linear projection $\pi\colon \mathbb{P}^3 \setminus L \to \mathbb{P}^1$, and a typical fiber of $\pi\colon S \to \mathbb{P}^1$.