ALGEBRAIC GEOMETRY: HOMEWORK 6

This homework is due on Friday, September 27 by 5pm.

(1) Show that $\mathbb{P}^1 \times \mathbb{A}^1$ is neither affine nor projective. What is the ring of regular functions on this variety?

(2) Let $Z \subset \mathbb{P}^n$ be a projective variety, and $X \subset \mathbb{P}^n \times \mathbb{A}^m$ a closed set. For $t \in \mathbb{A}^m$, let $X_t \subset \mathbb{P}^n \times \{t\} = \mathbb{P}^n$ denote the fiber of X over t under the second projection $X \to \mathbb{A}^m$. Show that the set of $t \in \mathbb{A}^m$ such that $Z \subset X_t$ is Zariski closed.
Hint: For a given degree d, it may be useful to consider the vector subspace of $k[X_0, \dots, X_n]_d$ consisting of polynomials that vanish on Z.

(3) As usual, let us identify the set of $n \times n$ matrices with $\mathbb{A}^{n^2}$. Let $S \subset \mathbb{A}^{n^2} \times \mathbb{A}^{n^2}$ be the set of pairs of matrices (A, B) such that A and B have a common eigenvector. Prove that S is a Zariski closed subset of $\mathbb{A}^{n^2} \times \mathbb{A}^{n^2}$.
Translation: There exist polynomial equations involving the entries of two matrices that are satisfied if and only if the two matrices share an eigenvector!

Remark. Properties of dimension imply that for $n = 2$, *one* polynomial equation suffices. What is it? (Food for thought, not to be turned in!)