ALGEBRAIC GEOMETRY: HOMEWORK 9

This homework is due on Friday October 18 by 5pm. Feel free to use the principal ideal theorem and its consequences proved in class.

(1) Prove that $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are not isomorphic to each other. Do the same for the Fermat cubic surface S and $\mathbb{P}^2$ (assume char $k > 3$).
Challenge (not to be turned in): Do the same for S and $\mathbb{P}^1 \times \mathbb{P}^1$.

(2) The *Krull dimension* of a ring R is the largest n for which there exists a strictly increasing chain
$$\mathfrak{P}_0 \subset \cdots \subset \mathfrak{P}_n$$
of prime ideals of R.
  (a) What is the Krull dimension of $\mathbb{Z}$?
  (b) Let X be an irreducible affine variety. Prove that the Krull dimension of $k[X]$ is equal to the dimension of X.

(3) Some time ago, we saw the geometric meaning of idempotents of a ring. Here is the analogue for zero-divisors. Let X be an affine variety (not necessarily irreducible). Show that $f \in k[X]$ is a zero-divisor if and only if f vanishes identically on some irreducible component of X. (Recall that an element a of a ring is a zero-divisor if there exists a non-zero element b such that $ab = 0$ in the ring.)