ALGEBRAIC GEOMETRY: HOMEWORK 10

This homework is due on Friday October 25 by 5pm.

Let $V = k[X, Y, Z]_d$, the vector space of homogeneous polynomials in three variables of degree d. Let $\Delta \subset \mathbb{P}V$ be the set F that define singular curves (to be defined more precisely soon). In this sequence of exercises, we will prove that Δ is a closed subvariety, irreducible of codimension 1.

(1) Prove the identity ("Euler identity"):

$$d \cdot F = X\frac{\partial F}{\partial X} + Y\frac{\partial F}{\partial Y} + Z\frac{\partial F}{\partial Z}.$$

Definition. We say that F is *singular* at p if F and all three partials of F vanish at p. Let $\Delta \subset \mathbb{P}V$ be the set of F that are singular somewhere.

(2) Let $D \subset \mathbb{P}V \times \mathbb{P}^2$ be the set of (F, p) such that F is singular at p. Show that D is a closed subvariety. Using the map $D \to \mathbb{P}^2$, prove that D is irreducible and find its dimension.
(3) Using the map $D \to \mathbb{P}V$, prove that Δ is a closed subvariety, and is irreducible of codimension 1.
(4) Does the map $D \to \mathbb{P}V$ have any positive dimensional fibers?
(5) (Not to be turned in). Since Δ is irreducible of codimension 1, it is given by the vanishing of *one* homogeneous polynomial on $\mathbb{P}V$. What is the degree of this polynomial?

Remark. The hypersurface $\Delta \subset \mathbb{P}V$ is called the *discriminant* hypersurface.

Remark. Working with three variables X, Y, Z is not important; everything works with n variables.