1 Regular functions and maps 1

Throughout this section, k is an algebraically closed field.

1.1 Regular functions WEEK3

Let $S \subset \mathbb{A}^n$ be a set and let $f\colon S \to k$ be a function. Let a be a point of S.

1.1.1 Definition (Regular function) We say that f is *regular* (or *algebraic*) *at* a if there exists a Zariski open set $U \subset \mathbb{A}^n$ and polynomials $p, q \in k[x_1, \ldots, x_n]$ with $q(a) \neq 0$ such that

$$f \equiv p/q \text{ on } S \cap U.$$

We say that f is *regular* if it is regular at all points of S.

In other words, f is regular at a point a if locally around a (in the Zariski topology), f can be expessed as a ratio of two polynomials. Although the definition of a regular function makes sense for $S \subset \mathbb{A}^n$, we use it only in the context of quasi-affine varieties.

1.1.2 Examples

1. A constant function is regular.
2. Every polynomial function is regular.
3. Sums and products of regular functions are regular. So, the set of regular functions forms a ring. This ring contains a copy of k, namely the constant functions.

1.1.3 Definition (Ring of regular functions) We denote the ring of regular functions on S by $k[S]$. This ring is a k-algebra.

1.1.4 Proposition (Local nature of regularity) Let f be a function on S, and let $\{U_i\}$ be an open cover of S. If the restriction of f to each U_i is regular, then f is regular.

Proof. — (1)

1.2 Regular functions on an affine variety WEEK3

It turns out that regular functions on closed subsets of $\mathbb{A}^n$ are just the polynomial functions! So, not only is there a global algebraic expression, we don't even need denominators.

1.2.1 Proposition Let $X \subset \mathbb{A}^n$ be a Zariski closed subset. Let f be a regular function on X. Then there exists a polynomial $P \in k[x_1, \ldots, x_n]$ such that $P(x) = f(x)$ for all $x \in X$.

Proof. By definition, we know that for every $x \in X$, there is a Zariski open set $U \subset X$ and polynomials p, q such that $f = p/q$ on U. The set U and the polynomials p, q may depend on x, so let us denote them by U_x, p_x, and q_x. We need to combine all of these p's and q's and construct a single polynomial P that agrees with f for all x.

This is done by a "partition of unity" argument. First, let us do some preparation. We know that $p_x/q_x = f$ on U_x, but we know nothing about p_x and q_x on the complement of U_x. Our first step is a small trick that lets us assume that both p_x and q_x are identically zero on the complement of U_x.

Since $U_x \subset X$ is open, its complement is closed. By the definition of the Zariski topology, this means that

$$X \setminus U_x = X \cap V(A),$$

for some $A \subset k[x_1, \ldots, x_n]$. Since $x \in U_x$, at least one of the polynomials in A must be non-zero at x. Let g be such a polynomial, and set $U'_x = X \cap \{g \neq 0\}$. Then $U'_x \subset U_x$ is a possibly smaller open set containing x. Set $p'_x = p_x \cdot g$ and $q'_x = q_x \cdot g$. Then we have $f = p'_x/q'_x$ on U'_x, and we also have $p'_x \equiv q'_x \equiv 0$ on $X \setminus U'_x$. So, we may assume from the beginning that both p_x and q_x are identically zero on the complement of U_x..

Now comes the crux of the argument. Suppose $X = V(I)$. Consider the set of "denominators" $\{q_x \mid x \in X\}$. Note that the system of equations

$$g = 0 \text{ for all } g \in I \text{ and } q_x = 0 \text{ for all } x \in X$$

has no solution!

Why is this true? — (2)

By the Nullstellensatz, this means that the ideal $I + \langle q_x \mid q \in X \rangle$ is the unit ideal. That is, we can write

$$1 = g + r_1 q_{x_1} + \cdots + r_m q_{x_m}$$

for some polynomials $r_1, \ldots, r_m$. Take $P = r_1 p_{x_1} + \cdots + r_m p_{x_m}$. Then $f = P$ on all of X.

Check the last equality. — (3)

□

—- Let $X \subset \mathbb{A}^n$ be any subset. We have a ring homomorphism

$$\pi \colon k[x_1, \ldots, x_n] \to k[X],$$

where a polynomial f is sent to the regular function it defines on X.

1.2.2 Proposition (Ring of regular functions of an affine) Let $X \subset \mathbb{A}^n$ be a closed subset. Then the ring homomorphism $\pi\colon k[x_1, \ldots, x_n] \to k[X]$ induces an isomorphism

$$k[x_1, \ldots, x_n]/I(X) \xrightarrow{\sim} k[X].$$

Proof. The map π is surjective by Proposition 1.2.1 and its kernel is $I(X)$ by definition. The result follows by the isomorphism theorems. □

1.3 Regular maps

WEEK3

Consider $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ and a function $f\colon X \to Y$. Write f in coordinates as

$$f = (f_1, \ldots, f_m).$$

1.3.1 Definition (Regular map) We say that f is *regular at a point* $a \in X$ if all its coordinate functions $f_1, \ldots, f_m$ are regular at a. If f is regular at all points of X, then we say that it is *regular*.

1.3.2 Example (Maps to $\mathbb{A}^1$) A regular map to $\mathbb{A}^1$ is the same as a regular function.

1.3.3 Example (An isomorphism) Let $U = \mathbb{A}^1 \setminus \{0\}$ and $V = V(xy - 1) \subset \mathbb{A}^2$. We have a regular function $\phi\colon V \to U$ given by $\phi(x, y) = x$. We have a regular function $\psi\colon U \to V$ given by $\psi(t) = (t, 1/t)$. These functions are mutual inverses, and hence we have a (bi-regular) isomorphism $U \cong V$.

1.4 Properties of regular maps

WEEK3

1.4.1 Proposition (Elementary properties of regular maps)

1. The identity map is regular.
2. The composition of two regular maps is regular.
3. Regular maps are continuous (in the Zariski topology).

Proof. The identity map is given by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n)$; each coordinate is a polynomial, and hence regular. The statement for composition is true because the composition of fractions of polynomials is also a fraction of polynomials. The third statement is left as homework. □

1.4.2 Proposition (Regular maps preserve regular functions) Let $\phi\colon X \to Y$ be a regular map. If f is a regular function on Y, then $f \circ \phi$ is a regular function on X.

Proof. View a regular function as a regular map to $\mathbb{A}^1$. Then this becomes a special case of composition of regular maps. □

— As a result, we get a k-algebra homomorphism $k[Y] \to k[X]$, often denoted by ϕ^*:

$$\phi^*(f) = f \circ \phi.$$

We thus get a (contravariant) functor from the category of (quasi-affine) varieties to k-algebras. On objects, it maps X to $k[X]$. On morphisms, it maps $\phi\colon X \to Y$ to $\phi^*\colon Y \to X$. It is easy to check that this recipe respects composition. That is, if we have maps $\phi\colon X \to Y$ and $\psi\colon Y \to Z$, and if we let $\psi \circ \phi\colon X \to Z$ be the composite, then

$$(\psi \circ \phi)^* = \phi^* \circ \psi^*.$$

1.4.3 Corollary (Isomorphic varieties have isomorphic rings of functions) If $\phi\colon X \to Y$ is an isomorphism of varieties, then $\phi^*\colon k[Y] \to k[X]$ is an isomorphism of k-algebras.

Proof. Let $\psi\colon Y \to X$ be the inverse of ϕ. Then $\psi^*\colon k[X] \to k[Y]$ is the inverse of ϕ^*. □

1.4.4 Proposition (For affines, map between rings induces map between spaces) Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be Zariski closed, and let $f\colon k[Y] \to k[X]$ be a homomorphism of k-algebras. Then there is a unique (regular) map $\phi\colon X \to Y$ such that $f = \phi^*$.

Proof. We know that $k[X] = k[x_1, \dots, x_n]/I(X)$ and $k[Y] = k[y_1, \dots, y_m]/I(Y)$. Let $\phi_i = f(y_i) \in k[X]$. Consider $\phi\colon X \to \mathbb{A}^m$ given by $\phi = (\phi_1, \dots, \phi_m)$. Then ϕ sends X to Y and is the unique map satisfying the required properties. □

> Prove the last statement. — (4)

1.4.5 Example (Bijection but not an isomorphism) Let $X = \mathbb{A}^1_k$ and $Y = V(y^2 - x^3) \subset \mathbb{A}^2_k$. We have a regular map $f\colon X \to Y$ given by $f(t) = (t^2, t^3)$. It is easy to check that f is a bijection, but not an isomorphism.

> Why is this not an isomorphism? — (5)

1.4.6 Example (Distinguished affine opens) Let $U_f \subset \mathbb{A}^n$ be the complement of $V(f)$. Then U_f is isomorphic to an affine variety, namely the variety $V(yf - 1) \subset \mathbb{A}^{n+1}$, where y denotes the $(n+1)$-th coordinate.

> Prove this. — (6)

1.4.7 Caution (Not all opens are affine) The previous proposition only applies to the complement of $V(f)$ for a single f! The complement of $V(I)$, in general, is not isomorphic to an affine variety. For example, the complement of the origin in $\mathbb{A}^2$ is not isomorphic to an affine variety.