1 Algebraic varieties

1.1 Definition **WEEK4**

The varieties we have seen so far have been sub-sets of the affine space. Using these as buildig blocks, we can construct general algebraic varieties. The definition is analogous to the definition of a manifold in differential geometry, using open subsets of $\mathbb{R}^n$ as building blocks.

Let X be a topological space. A *quasi-affine chart* on X consists of an open subset $U \subset X$, a quasi-affine variety V and a homeomorphism $\phi_{UV} : U \to V$. Via this isomorphism, we can "transport" the algebraic structure (for example, the notion of a regular function) from V to U.

Let $\phi_1 \colon U_1 \to V_1$ and $\phi_2 \colon U_2 \to V_2$ be two quasi-affine charts on X (see Figure 1). Set $U_{12} = U_1 \cap U_2$. Consider the open subsets $V_{12} = \phi_1(U_{12}) \subset V_1$ and $V_{21} = \phi_2(U_{12}) \subset V_2$. Being open subsets of quasi-affine varieties, they are themselves quasi-affine varieties. Furthermore, the map

$$\phi_2 \circ \phi_1^{-1} \colon V_{12} \to V_{21}$$

is a homeomorphism. We say that the two charts are *compatible* if this map is a (bi-regular) isomorphism.

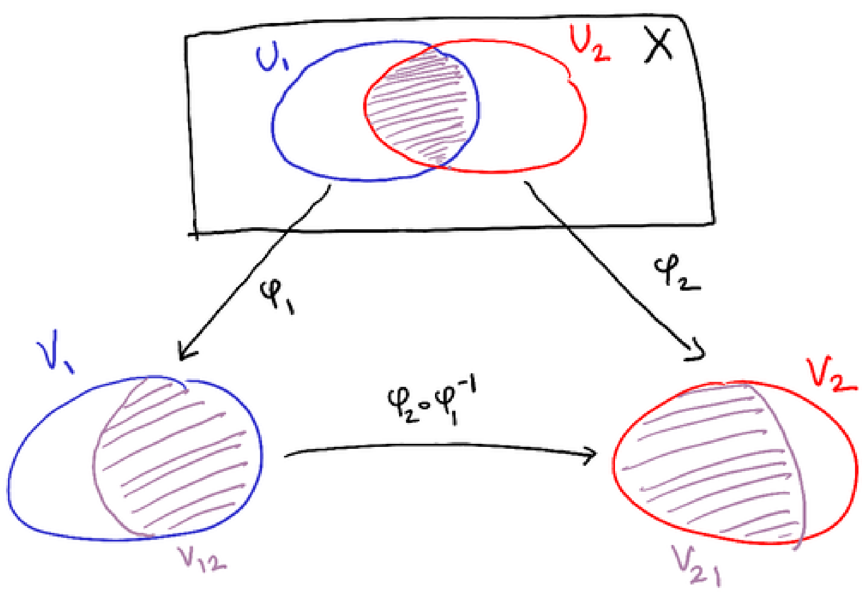

Figure 1: Compatible charts

When we have two charts, one on U_1 and another on U_2, then the intersection $U_1 \cap U_2$

gets two different charts. Compatibility ensures that these two charts are related by a bi-regular isomorphism, so that the algebraic structure coming from one is the same as the one coming from the other.

A *quasi-affine atlas* on X is a collection of compatible charts $\phi_i \colon U_i \to V_i$ such that the U_i cover X.

1.1.1 Definition (Algebraic variety) An *algebraic variety* is a topological space with a quasi-affine atlas.

1.1.2 Example (Quasi-affine varieties) A quasi-affine variety X is itself an algebraic variety. The atlas is the obvious one, consisting of the single chart $\mathrm{id} \colon X \to X$.

1.2 Projective spaces **WEEK4**

A fundamental example of an algebraic variety is the projective space.

1.2.1 Definition (Projective space) The *projective n-space over a field k*, denoted by $\mathbb{P}^n_k$, is the set of one-dimensional subspaces of k^{n+1}.

1.2.2 Intuition Before describing how $\mathbb{P}^n_k$ is an algebraic variety, let us build some intuition about projective space. For easy visualisations, it helps to take $k = \mathbb{R}$ or $k = \mathbb{C}$. A one dimensional subspace of k^{n+1} is also called a *line*. Note that, by this definition, a line must contain the origin.

Let us take $n = 0$. Then there is a unique one-dimenional subspace of $k^{n+1} = k$, so $\mathbb{P}^0_k$ is just a single point.

Let us take $n = 1$. Then $\mathbb{P}^1_k$ is the set of lines (through the origin) in k^2. Let us take $k = \mathbb{R}$. Every line through the origin is uniquely determined by its slope, which can be any element of $\mathbb{R}$, so it seems like $\mathbb{P}^1_{\mathbb{R}}$ is just a copy of $\mathbb{R}$. But the vertical line does not have a (finite) slope, so $\mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. In other words, $\mathbb{P}^1$ contains the usual real line, plus "a point at infinity".

It can be more instructive to see this in a picture. Fix a horizontal line L at, say, $y = -1$. Every line through the origin intersects L at a unique point, except the horizontal line. So if we discard the one point of $\mathbb{P}^1_k$ corresponding to the horizontal line, the rest is just a copy of L. If we had chosen a different reference line L, for example, a vertical one, then we get a similar description of $\mathbb{P}^1$ away from a single point. In fact, we can discard *any* one point of $\mathbb{P}^1$, and the rest will be a copy of $\mathbb{R}$.

Let us take $n = 2$. Then $\mathbb{P}^2_k$ is the set of lines (through the origin) in k^3. We can use the same technique as before: fix a reference plane P at $z = -1$. Then most lines are uniquely characterised by their intersection point with P. The only exceptions are the lines parallel to $z = -1$, that is, the lines lying in the plane $z = 0$, which we miss. But these form a small projective space $\mathbb{P}^1$. So we see that $\mathbb{P}^2 = P \sqcup \mathbb{P}^1$.

1.2.3 Topology

A one-dimensional subspace of k^{n+1} is spanned by a non-zero vector $(a_0, \dots, a_n)$. Two vectors $(a_0, \dots, a_n)$ and $(b_0, \dots, b_n)$ span the same subspace if and only if there exists $\lambda \in k^\times$ such that

$$(b_0, \dots, b_n) = (\lambda a_0, \dots, \lambda a_n).$$

So, we can identify $\mathbb{P}^n$ with the equivalence classes of non-zero vectors $(a_0, \dots, a_n)$ where two non-zero vectors are considered equivalent if one is a scalar multiple of the other. In other words, we have

$$\mathbb{P}^n_k = (\mathbb{A}^{n+1} \setminus 0)/\text{scaling}.$$

We denote the equivalence class of $(a_0, \dots, a_n)$ by $[a_0 : \dots : a_n]$.

We give $\mathbb{P}^n_k$ the quotient topology inherited from $\mathbb{A}^{n+1} \setminus 0$. That is, a set $U \subset \mathbb{P}^n_k$ is open/closed if and only if its pre-image in $\mathbb{A}^{n+1} \setminus 0$ is open/closed.

For example, consider the subset U_n of $\mathbb{P}^n_k$ consisting of $[a_0 : \dots : a_n]$ with $a_n \neq 0$. Its preimage in the set of $(a_0, \dots, a_n) \in \mathbb{A}^{n+1} \setminus 0$ with $a_n \neq 0$, which is a (Zariski) open set. Hence U_n is open in $\mathbb{P}^n_k$. Likewise, $U_0, U_1, \dots$ are also open. Note that we have

$$\mathbb{P}^n_k = U_0 \cup \dots \cup U_n;$$

that is, the sets $U_0, \dots, U_n$ form an open cover of $\mathbb{P}^n$.

Consider a point $[a_0 : \dots : a_n] \in U_0$, so that $a_0 \neq 0$. By scaling by $\lambda = a_0^{-1}$, we have a distinguished representative of this point of the form $[1 : b_1 : \dots : b_n]$, which we can think of as a point $(b_1, \dots, b_n) \in \mathbb{A}^n$. Thus, we have a bijection $\phi_0 \colon U_0 \to \mathbb{A}^n$, and similarly $\phi_1 U_i \to \mathbb{A}^n$.

1.2.4 Proposition (Charts of the projective space)

1. The bijections $\phi_i \colon U_i \to \mathbb{A}^n$ defined above are homeomorphisms.
2. The charts $\phi_i \colon U_i \to \mathbb{A}^n$ are mutually compatible, and hence give an atlas on $\mathbb{P}^n$.

1. This is not obvious, also not hard, but also not very enlightening. Let us skip this.
2. Do this! — (1)

1.2.5 Open and closed subvarieties

Let X be an algebraic variety, and $Y \subset X$ an open or closed subset. Then Y inherits the structure of an algebraic variety. To get, the atlas for Y, let $\phi_i \colon U_i \to V_i$ be an atlas for X. For Y, we just take $\phi_i \colon U_i \cap Y \to \phi(U_i \cap Y)$.

Explain why this is an atlas for Y — (2)

1.2.6 Proposition (Closed subvarieties of projective space 1) Let $F \in k[X_0, \dots, X_n]$ be a homogeneous polynomial. Let $V(F) \subset \mathbb{P}^n$ be the set of points $\{[a_0 : \cdots : a_n] \mid F(a_0, \dots, a_n) = 0\}$. Then $V(F)$ is a closed subset.

Explain why $V(F)$ is well-defined (that is, the condition $F(a_0, \dots, a_n) = 0$ does not depend on the chosen representative of the equivalence class). Then explain why $V(F)$ is closed. — (3)

1.2.7 Proposition (Closed subvarieties of projective space 2) Let $I \subset k[X_0, \dots, X_n]$ be a homogeneous ideal.

Define $V(I) \subset \mathbb{P}^n$ and show that it is a closed subset. — (4)

1.2.8 Proposition (Closed subvarieties of projective space 3) Conversely, let $X \subset \mathbb{P}^n$ be a closed subset. Then there exists a homogeneous ideal $I \subset k[X_0, \dots, X_n]$ such that $X = V(I)$.

Proof. Assume that X is non-empty. Let $\pi \colon \mathbb{A}^{n+1} \setminus 0 \to \mathbb{P}^n$ be the quotient map. Let $C \subset \mathbb{A}^n$ be the closure of $\pi^{-1}(X)$.

Prove that C is conical, that is, if $x \in C$ then $\lambda x \in C$ for every scalar $\lambda \in k$. Conclude using Homework 1 that $C = V(I)$ for a homogeneous ideal I. Prove that $X = V(I)$ in $\mathbb{P}^n$. — (5)

□

1.2.9 Example (Linear subspaces) Suppose $I \subset k[X_0, \dots, X_n]$ is generated by (homogeneous) linear equations. Then $V(I) \subset \mathbb{A}^{n+1}$ is a sub-vector space $W \subset \mathbb{A}^{n+1}$, and $V(I) \subset \mathbb{P}^n$ is naturally the projective space of W. We call such $V(I) \subset \mathbb{P}^n$ *linear subspaces*, or "lines", "planes", etc. See that any two distinct lines in $\mathbb{P}^2$ intersect at a unique point, and through any two distinct points in $\mathbb{P}^2$ passes a unique line.