1 Regular functions and regular maps 2

1.1 Regular functions and maps **WEEK5**

1.1.1 Proposition (regularity does not depend on the chart) Let X be an algebraic variety and $f\colon X \to k$ a function. Let $\phi_1\colon U_1 \to V_1$ and $\phi_2\colon U_2 \to V_2$ be two compatible charts such that x lies in both U_1 and U_2. Denote the images of x in the two charts by v_1 and v_2. Consider the functions $f \circ \phi_1^{-1}\colon V_1 \to k$ and $f \circ \phi_2^{-1}\colon V_2 \to k$. Then the first is regular at v_1 if and only if the second is regular at v_2.

Prove this. — (1)

1.1.2 Definition (regular function on a variety) Let $f\colon X \to k$ be a continuous function. We say that f is regular at x if for some (equivalently, for every) chart $\phi\colon U \to V$ with $x \in U$, the function $f \circ \phi^{-1}\colon V \to k$ is regular at $\phi(x)$. We say that f is regular on X if it is regular at all points $x \in X$.

1.1.3 Definition (regular map between varieties) Let X and Y be algebraic varieties and $f\colon X \to Y$ a continuous map. We say that f is regular at a point $x \in X$ if for any (equivalently, for every) chart $\phi\colon U \to V$ with $x \in U$ and $\psi\colon U' \to V'$ with $f(x) \in U'$, the composite map

$$\psi \circ f \circ \phi^{-1}\colon V \dashrightarrow V'$$

is regular at $\phi(x)$.T The reason for the dashed arrow is that the domain of $\psi \circ f \circ \phi^{-1}$ may not be all of V, but only an open subset of V. To be precise, the domain is $\phi(U \cap f^{-1}(U'))$. But the domain contains $\phi(x)$, so it makes sense to talk about the regularity at $\phi(x)$.

See Figure 1 for a picture (the bottom arrow should be dashed).

1.2 Examples **WEEK5**

For quasi-affine varieties, these definitions do not add anything new.

1.2.1 Example Let $X = \mathbb{P}^1$. Set $f([X:Y]) = X/Y$. Then f is defined at all points except the point $[1:0]$, and is a regular function on $\mathbb{P}^1 \setminus \{[1:0]\}$. More generally, let $X = \mathbb{P}^n$ and let $F, G \in k[X_0, \dots, X_n]$ be homogeneous polynomials of the same degree. The function

$$[X_0 : \cdots : X_n] \mapsto F(X_0, \dots, X_n)/G(X_0, \dots, X_n)$$

is regular outside $V(G)$.

Prove this. —- (2)

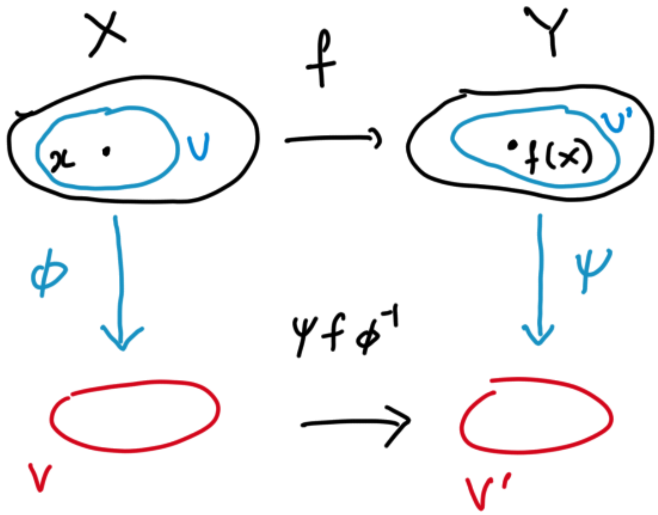

Figure 1: A map is regular if it is regular with respect to the charts.

1.2.2 Example Let $X = \mathbb{P}^n$ and let $F_0, \ldots, F_m$ be homogeneous polynomials of the same degree. Let $Z \subset \mathbb{P}^n$ be $V(F_0, \ldots, F_m)$. Then the formula

$$[X_0 : \cdots : X_n] \mapsto [F_0(X_0, \ldots, X_n) : \cdots : F_m(X_0, \ldots, X_n)]$$

defines a regular map from $X \setminus Z$ to $\mathbb{P}^m$.

Prove this. —- (3)

1.2.3 Example The natural map $\mathbb{A}^{n+1} - 0 \to \mathbb{P}^n$ is regular.

1.2.4 Example (Automorphisms of $\mathbb{P}^n$) Consider the $n+1$-dimensional k-vector space V spanned by $X_0, \ldots, X_n$. Pick any basis $\ell_0, \ldots, \ell_n$ of this vector space. Then we have a regular map

$$L \colon \mathbb{P}^n \to \mathbb{P}^n$$
$$[X_0 : \cdots : X_n] \mapsto [\ell_0 : \cdots : \ell_n].$$

Explicitly, if we write

$$\ell_i = L_{i,0}X_0 + \cdots + L_{i,n}X_n$$

and write our homogenous vector as a column vector, then the map is

$$[X] \mapsto [LX].$$

In other words, it is induced by the invertible linear map $L\colon V \to V$. As a result, it has an inverse, induced by the inverse of the matrix M:

$$[X] \mapsto [MX].$$

In this way, we get an action of $GL_n(k)$ on $\mathbb{P}^n$. But notice that a matrix L and a scalar multiple λL induce the same map on $\mathbb{P}^n$. So the action descends to an action of the group $PGL_n(k) = GL_n(k)/\text{scalars}$.

1.2.5 Example (regular functions on $\mathbb{P}^1$) The previous example gave examples of regular functions on (strict) open subsets of the projective space. It turns out that there are *no* regular functions on $\mathbb{P}^n$ other than the constant functions!

> Prove this for $n = 1$. Then deduce it for all n using that through any two distinct points in $\mathbb{P}^n$ passes a projective line. — (4)

1.3 Elementary properties of regular maps **WEEK5**

1.3.1 Proposition The identity map is regular. The composition of two regular maps is regular.

1.4 The Veronese embedding **WEEK5**

Let $n \geq 1$, and consider the k-vector space of degree n homogeneous polynomials in X, Y. This vector space has dimension $n+1$. Choose a basis, for example, let us take $X^n, X^{n-1}Y, \ldots, XY^{n-1}, Y^n$. Then we have a regular map

$$v_n\colon \mathbb{P}^1 \to \mathbb{P}^n$$
$$[X : Y] \mapsto [X^n : \cdots : Y^n].$$

1.4.1 Proposition (Veronese curves) The image of v_n is a closed subset of C of $\mathbb{P}^n$. If we denote the homogeneous coordinates on $\mathbb{P}^n$ by $[U_0 : \cdots : U_n]$, then C is cut out by the equations

$$\{U_iU_j - U_kU_\ell \mid 0 \leq i, j, k, l \leq n \text{ and } i + j = k + \ell.$$

> Prove this. — (5)

1.4.2 Proposition (Veronese curves continued) The map $v_n\colon \mathbb{P}^1 \to C$ is in fact an isomorphism.

Define the inverse map. — (6)

The proposition above generalises to all dimensions. Consider the k-vector space of degree n homogeneous polynomials in $X_0, \dots, X_m$. It has dimension $N = \binom{n+m}{m}$. Choosing a basis gives a map $\mathbb{P}^m \to \mathbb{P}^N$. The image of this map is a closed subvariety Z and the map $\mathbb{P}^m \to Z$ is an isomorphism. The equations of Z and the description of the inverse map are analogous to the $m = 1$ case, but (understandably) somewhat more cumbersome.

1.5 Example: Conics in $\mathbb{P}^2$ **WEEK5**

The 2-nd Veronese embedding maps $\mathbb{P}^1$ isomorphically onto the zero-locus of a degree 2 equation in $\mathbb{P}^2$. More explicitly, the image of the map

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^2 \\ [X : Y] &\mapsto [X^2 : XY : Y^2] \end{aligned}$$

is the set $V(UW - V^2)$. Now recall a theorem from linear algebra. You may have proved this only over $\mathbb{C}$ or even over $\mathbb{R}$ (in which case, there are some signs you have to reckon with), but the same proof works for all algebraically closed fields of characteristic $\neq 2$.

1.5.1 Theorem (quadratic forms) Let k be an algebraically closed field of characteristic $\neq 2$ and let q be a quadratic form on a k-vector space V. Then there exists a basis $X_0, \dots, X_n$ for V such that

$$q(X_0, \dots, X_n) = X_0^2 + \cdots + X_\ell^2.$$

The form is called non-degenerate if $\ell = n$.

1.5.2 Corollary Let Q be a non-degenate conic in $\mathbb{P}^2$. Then Q is isomorphic to $\mathbb{P}^1$.

Proof. All non-degenerate conics are isomorphic to each other, and we know that at least one of them—the 2nd Veronese image of $\mathbb{P}^1$—is isomorphic to $\mathbb{P}^1$. □

1.5.3 Question What do the degenerate conics in $\mathbb{P}^2$ look like?