1 Products and the Segre embedding

1.1 Definition of the product variety WEEK6

If X and Y are algebraic varieties, then their product set $X \times Y$ is naturally an algebraic variety. This, in theory, should be completely straightforward (and it is), but you have to be slightly careful because the Zariski topology of $X \times Y$ is *not* the product topology.

First, suppose $X = \mathbb{A}^m$ and $Y = \mathbb{A}^n$, then $X \times Y = \mathbb{A}^{m+n}$ is an algebraic variety. Observe that the Zariski topology on $\mathbb{A}^{m+n}$ is *not* the product topology.

Second, if $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^m$ are both closed (or open), then $X \times Y \subset \mathbb{A}^{m+n}$ is closed (or open), so it is naturally an algebraic variety.

Prove that products of closed (or open) are closed (or open). — (1)

Third, by combining the cases of closed/open and taking intersections, we get that if X and Y are locally closed, then $X \times Y \subset \mathbb{A}^{m+n}$ is also locally closed, and hence an algebraic variety. So the case of quasi-affine varieties is done.

In general, suppose X has the quasi-affine atlas $\{\phi_i \colon U_i \to V_i\}$ and Y has the quasi-affine atlas $\{\phi'_j \colon U'_j \to V'_j\}$. Then the product $X \times Y$ is covered by the sets $U_i \times U'_j$. We *declare* the product map $U_i \times U'_j \to V_i \times V'_j$ to be a homeomorphism; that is, we give $U_i \times U'_j$ the Zariski topology of $V_i \times V'_j$. Then, we declare a set $Z \subset X \times Y$ to be closed (or open) if and only if for all i, j, the intersection $Z \cap U_i \times U'_j$ is closed (or open) in $U_i \times U'_j$. It is easy to check that this gives $X \times Y$ a topology under which $U_i \times U'_j$ is an open cover, and the maps

$$\phi_i \times \phi'_j \colon U_i \times U'_j \to V_i \times V'_j$$

are a family of compatible charts.

1.1.1 Proposition The two projection maps $X \times Y \to X$ and $X \times Y \to Y$ are regular. A map $\phi \colon Z \to X \times Y$ is regular if and only if the two component maps $\phi_1 \colon Z \to X$ and $\phi_2 \colon Z \to Y$ are regular.

Proof. Skipped (for being easy). □

1.1.2 Remark If you have seen some category theory (in particular, Yoneda's lemma), you will see that the above proposition characterises the product "uniquely up to a unique isomorphism."

1.2 Example WEEK6

Write down the charts of $\mathbb{P}^1 \times \mathbb{P}^1$, and the transition function between one pair of charts. — (2)

1.3 Closed subsets of $\mathbb{P}^n \times \mathbb{P}^m$ WEEK6

Let $F \subset k[X_0, \ldots, X_n, Y_0, \ldots, Y_m]$ be a bi-homogeneous polynomial of bi-degree (a, b). This means that every term in F has X-degree a and Y-degree b. Or equivalently, for any $\lambda, \mu \in k$, we have

$$F(\lambda X_0, \ldots, \lambda X_n, \mu Y_0, \ldots, \mu Y_m) = \lambda^a \mu^b F(X_0, \ldots, X_n, Y_0, \ldots, Y_m).$$

Then $V(F) \subset \mathbb{P}^n \times \mathbb{P}^m$ is well-defined and is a closed subset. Same story for bi-homogeneous ideals.

1.4 The Segre embedding WEEK6

The Segre embedding is a closed embedding of $\mathbb{P}^n \times \mathbb{P}^m$ in a bigger projective space. It is a cool example, but it is also of theoretical importance. The most studied and the most well-behaved varieties are projective varieties (varieties isomorphic to closed subsets of projective space) or somewhat more generally quasi-projective varieties (varieties isomorphic to locally closed subsets of projective space). The Segre embedding shows that this class of varieties is closed under products.

Let $N = (m+1)(n+1) - 1$. Consider the Segre map $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ defined by

$$[X_0, \ldots, X_n], [Y_0, \ldots, Y_m] \mapsto [X_i \cdot Y_j].$$

It is easy to check that this map is regular.

A good way to think about this map is as follows. Think of elements of $\mathbb{P}^n$ as row vectors up to scaling, $\mathbb{P}^m$ as column vectors (up to scaling), and $\mathbb{P}^n$ as $(n+1) \times (m+1)$-matrices up to scaling. Then the product XY of $X \in \mathbb{P}^n$ and $Y \in \mathbb{P}^m$ is an $(n+1) \times (m+1)$ matrix, which taken up to scaling, defines an element of $\mathbb{P}^N$. Observe that matrix XY has rank 1, and hence the Segre map lands in the subspace $Z \subset \mathbb{P}^N$ corresponding to matrices of rank 1.

Now, a rank 1 matrix can be written as a product XY, and up to scaling, such an expression is unique. As a result, the Segre map is a bijection from $\mathbb{P}^n \times \mathbb{P}^m \to Z$. But more is true.

1.4.1 Theorem (Segre embedding) The rank 1 locus $Z \subset \mathbb{P}^N$ is closed, and the Segre map $\mathbb{P}^n \times \mathbb{P}^m \to Z$ is a bi-regular isomorphism.

Proof. Consider an $(n+1) \times (m+1)$ matrix M. Then M has rank 1 if and only if all 2×2 minors of M vanish. Hence, Z is the zero-locus of all 2×2-minors, which are homogeneous polynomials in the entries of the matrix.

To prove that the Segre map is an isomorphism onto Z, we must construct a regular inverse $Z \to \mathbb{P}^n \times \mathbb{P}^m$.

Do it! — (3).

□

1.4.2 Definition (Projective and quasi-projective varieties) A *projective variety* is a variety isomorphic to a closed subset of projective space. A *quasi-projective variety* is a variety isomorphic to an open subset of a projective variety.

1.4.3 Proposition (All quasi-affines are quasi-projective) Every quasi-affine variety is quasi-projective.

Proof. The affine space $\mathbb{A}^n$ is (isomorphic to) an open subset of $\mathbb{P}^n$. So a locally closed subset of $\mathbb{A}^n$ is also a locally closed subset of $\mathbb{P}^n$. □

1.4.4 Corollary (of the Segre embedding) If X and Y are (quasi)-projective, then so is $X \times Y$.

Proof. Suppose X and Y are projective, say $X \subset \mathbb{P}^n$ is closed and $Y \subset \mathbb{P}^m$ is closed. Then $X \times Y \subset \mathbb{P}^n \times \mathbb{P}^m$ is closed. The Segre embedding shows that $\mathbb{P}^n \times \mathbb{P}^m$ is isomorphic to a closed subset of $\mathbb{P}^N$. Hence $X \times Y$ is isomorphic to a closed subset of $\mathbb{P}^n$. In other words, $X \times Y$ is projective.

In general, suppose X (resp. Y) is an open subset of a projective variety $\overline{X}$ (resp. $\overline{Y}$). Then $X \times Y$ is an open subset of $\overline{X} \times \overline{Y}$, which we proved is projective. So $X \times Y$ is quasi-projective. □

1.4.5 Exercise (Quadric surfaces) The Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ lives in $\mathbb{P}^3$.

Describe the equations that cut out the image. Conclude that every non-degenerate quadric in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. — (4)

1.4.6 Exercise ($\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$)

Are $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ isomorphic? — (5) Use whatever tools you have over your favourite field to answer this.

1.4.7 The diagonal embedding Consider the diagonal map $\Delta\colon \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$. The image of Δ is a closed subset. If we use homogeneous coordinates $[X_0 : \cdots : X_n]$ and $[Y_0 : \cdots : Y_n]$ on the two copies of $\mathbb{P}^n$, then the image is the vanishing set of the bi-homogeneous polynomials

$$X_i Y_j - X_j Y_i \text{ for } 0 \le i, j \le n.$$

Algebraic varieties X for which the image of the diagonal map $\Delta\colon X \to X \times X$ is closed are called *separated*. This condition is analogous to the Hausdorff condition in topology. Not all varieties are separated, but all quasi-projective varieties are.

1.4.8 Proposition All quasi-projective varieties are separated.

Prove this. — (6)