1 Grassmannians **WEEK7**

Grassmannians are a natural generalisation of the projective space. In terms of ubiquity, they are only second in line after projective spaces. In other words, they are pretty important. Fix positive integers m and n with $m \leq n$.

1.0.1 Defintion (Grassmannian) The Grassmannian of m-planes in k^n, denoted by $\mathrm{Gr}(m,n)$ is the set of m-dimensional subspaces of k^n. In particular, $\mathrm{Gr}(1,n+1)$ is the projective space $\mathbb{P}^n$.

1.1 Topology

We endow $\mathrm{Gr}(m,n)$ with a topology by expressing it as a quotient. An m-plane in k^n is spanned by m linearly independent vectors $v_1,\dots,v_m$ in k^n. Two sets of vectors $v_1,\dots,v_m$ and $w_1,\dots,w_m$ span the same m-plane if and only if there exists an invertible $m\times m$-matrix A such that

$$(v_1,\dots,v_m)A = (w_1,\dots,w_m).$$

Let $U \subset (\mathbb{A}^n)^m = \mathbb{A}^{nm}$ denote the set of $(v_1,\dots,v_m)$ with $v_i \in \mathbb{A}^n$ such that $v_1,\dots,v_m$ are linearly independent. Then U is a Zariski open subset. We have an action of $\mathrm{GL}_m(k)$ on U by right-multiplication, and $\mathrm{Gr}(m,n)$ is the space of orbits. That is, we have

$$\mathrm{Gr}(m,n) = U/\mathrm{GL}_m(k).$$

We give $\mathrm{Gr}(m,n)$ the quotient topology.

1.2 Atlas

Let us write vectors in k^n as column vectors. Then we can write an m-tuple $(v_1,\dots,v_m)$ as an $n\times m$ matrix, say V. If $v_1,\dots,v_m$ are linearly independent, then V has rank m. That is, V contains an $m\times m$ sub-matrix that is invertible. Let $I \subset \{1,\dots,n\}$ be an m-element subset, and let V_I denote the $m\times m$ submatrix of V obtained by choosing the I-rows (see Figure). Let $U_I \subset \mathrm{Gr}(n,m)$ be the subset represented by the V for which V_I is invertible. Then U_I is an open subset. For every point v in U_I, we can choose a unique representative matrix V such that V_I is the identity matrix. (To do this, first pick any representative V and then multiply on the right by V_I^{-1}.) We get a bijection

$$\phi_I \colon U_I \to \mathbb{A}^{m(n-m)}$$

defined by the following formula

$$\phi_I(v) = V_{I^c},$$

where V is the unique matrix whose columns span v and which satiesfies $V_I = \mathrm{id}$. The notation V_{I^c} means take the rows of V corresponding to I^c—that is, drop the rows corresponding to I. See 1 for a picture.

1.2.1 Proposition The collection of charts $\{\phi_I\}$ gives an atlas on the Grassmannian.

> We need to prove that (a) the maps ϕ_I are homeomorphisms, and (b) the charts are compatible. We will skip (a). Prove (b) in an example, say $n = 4$ and $m = 2$, and you will understand the general argument. — (1)

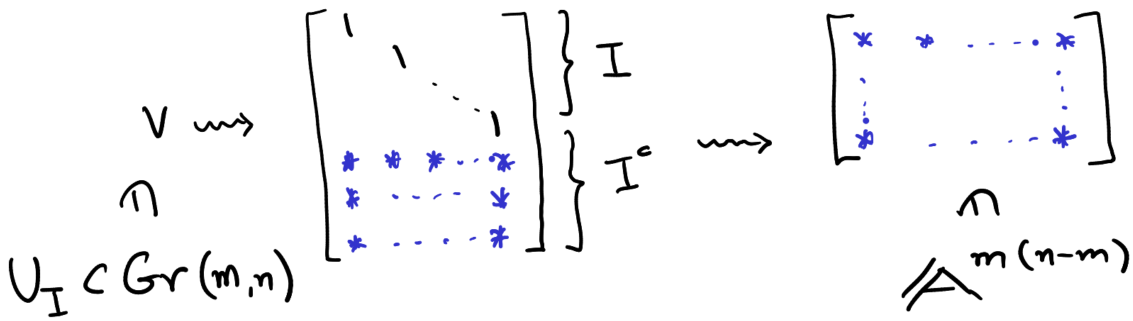

Figure 1: A chart of the Grassmannian

1.3 The Plucker embedding

There is a way to embed Grassmannians as closed subsets of projective spaces. In due course, we'll see that projective varieties (varieties isomorphic to closed subsets of projective spaces) are the best varieties, and the Plucker embedding shows that Grassmannians are in the club.

The map is simple. It goes

$$p\colon \mathrm{Gr}(m,n) \to \mathbb{P}^N,$$

where $N = \binom{n}{m} - 1$. Take an m-dimensional subspace v of k^n represented by an $n \times m$ matrix V. Define

$$p(v) = [\det V_I],$$

where I ranges over all m-element subsets of $1, \dots, n$. This is well-defined. First of all, not all determinants are 0, because V has rank m. Secondly, a different representative of v has the form VA, where A is an invertible $m \times m$ matrix, but then all the determinants are multiplied by the same scalar, namely $\det A$.

Show that the Plucker map p is regular. — (2)

1.3.1 Proposition The image of the Plucker embedding p is a closed subset of $\mathbb{P}^N$ and the map p is an isomorphism onto the image.

Proof. It is not so easy to identify the homogeneous polynomials that cut out the image. It is easier to work on charts.

Let represent the homogeneous coordinates of $\mathbb{P}^N$ by $[X_I]$, where I ranges over m-element subsets I of $\{1, \dots, n\}$. Let $U_I \subset \mathbb{P}^N$ be the standard open set; the one where $X_I \neq 0$.

Let Z be the image of p. To show that Z is closed, it is enough to show that $Z \cap U_I$ is a closed subset of U_I for each I. Then, to show that p is an isomorphism onto its image, it is enough to construct a regular map

$$Z \cap U_I \to p^{-1}(U_I)$$

which is an inverse to p. □

Do it for one I in the case $n = 4$ and $m = 2$, and you will understand the general argument.