1 Local rings and tangent spaces

Let X be an algebraic variety and $x \in X$ a point. Let us describe a construction that lets us study the geometry of X near x using algebra. We will construct a ring $O_{X,x}$ called the local ring of X at x. This will be non-trivial even when X is not affine, and will contain all information about the local geometry of X near x.

1.1 The ring of germs

A *germ* of a regular function at x is an equivalence class of (U, f) where $U \subset X$ is an open set containing x and f is a regular function on U. Two pairs (U, f) and (V, g) are equivalent if there is an open set W containing x with $W \subset U$ and $W \subset V$ such that $f|_W = g|_W$.

The idea is that only the behaviour of the function near x matters. The idea is not unique to algebraic geometry; it is useful in any geometric context.

Let $O_{X,x}$ be the set of germs of regular functions at x. There is an obvious addition and multiplication of germs, which makes $O_{X,x}$ a ring and there is an obvious copy of k inside this ring, which makes it a k-algebra. Note that if $U \subset X$ is an open subset containing x, then $O_{X,x} = O_{U,x}$. The local ring gives a convenient language to talk about statements of the form "…. holds in some open set containing x" without being explicit about the open set. By abuse of notation, when we specify elements of $O_{X,x}$, we only specify the f and drop the U.

The definition of $O_{X,x}$ is very similar to the definition of rational functions (if X is irreducible), except that all the open sets in question are supposed to contain the point x. Here is the precise relationship.

1.1.1 Proposition (Connection with the fraction field) Let X be irreducible. Then we have a natural inclusion $O_{X,x} \subset k(X)$ and $O_{X,x}$ is the set of rational functions on X which are defined at x.

Proof. Skipped. □

In particular, if X is affine and irreducible, it is easy to calculate the ring of germs.

1.1.2 Proposition (Description for affines 1) Let X be irreducible and affine. Then the ring $O_{X,x} \subset \operatorname{frac} k[X]$ is given by

$$O_{X,x} = \left\{ \frac{f}{g} \mid f \in k[X], g \in k[X], g(x) \neq 0. \right\}$$

That is, in the denominator, we are only allowed to have functions which are not zero at x.

Proof. Skipped. □

Here is another explicit description of the local ring for an affine.

1.1.3 Proposition (Description for affines 2) Let $X \subset \mathbb{A}^n$ be the closed subset with $I(X) = \langle f_1, \dots, f_r \rangle$. Let $x = (a_1, \dots, a_n) \in X$. Then $O_{X,x}$ is the quotient of $O_{\mathbb{A}^n,x}$ by the ideal generated by $f_1, \dots, f_r$.

(1) — Prove this.

1.1.4 Functoriality The construction of the local ring is functorial. That is, if we have a regular map $f\colon X \to Y$ such that $y = f(x)$, then pull-back of functions induces a k-algebra homomorphism

$$f^*\colon O_{Y,y} \to O_{X,x}.$$

If f is a local isomorphism—that is, if there exist opens $U \subset X$ and $V \subset Y$ containing x and y, respectively, such that f induces an isomorphism $f\colon U \to V$—then f^* is an isomorphism.

Let $m \subset O_{X,x}$ be the set of germs f such that $f(x) = 0$. Equivalently, let m be the kernel of the map

$$O_{X,x} \to k$$

that sends f to $f(x)$. Then m is a maximal ideal. It is not hard to see that this is the *only* maximal ideal of $O_{X,x}$.

1.1.5 Proposition (Locality) The ring $O_{X,x}$ has a unique maximal ideal m, which consists of functions that vanish at x.

Proof. It is enough to show that every $f \in O_{X,x}$ with $f \notin m$ is a unit in $O_{X,x}$. But if $f \notin m$ then $f(x) \neq 0$, and hence f is invertible in some neighborhood of x. $\square$

A *local ring* is a ring with a unique maximal ideal. We just proved that $O_{X,x}$ is a local ring. Local rings are intensely studied in commutative algebra, mostly because they arise as rings of germs in geometry.

1.2 Tangent space

We will define the tangent space to X at x as the set of tangent vectors to X at x. There are many equivalent ways to think about tangent vectors.

1.2.1 Infinitesimal curves A *tangent vector* to X at x is a k-algebra homomorphism

$$v\colon O_{X,x} \to k[\epsilon]/\epsilon^2.$$

Let us understand this concretely when X is affine, say $X \subset \mathbb{A}^n$ closed. Let $I(X) = \langle f_1, \dots, f_r \rangle$. Then X is the set of k-valued solutions of the system of equations

$$f_1(x_1, \dots, x_n) = 0, \dots, f_r(x_1, \dots, x_n) = 0. \tag{1}$$

1.2.2 Proposition (Infinitesimal curves) Let $x = (a_1, \dots, a_n) \in X$. We have a bijection between k-algebra homomorphisms $O_{X,x} \to k[\epsilon]/\epsilon^2/$ and $k[\epsilon]/\epsilon^2$-valued solutions of the system (1) based at $(a_1, \dots, a_n)$, that is, solutions of the form $(a_1 + b_1\epsilon, \dots, a_n + b_n\epsilon)$.

To go from a homomorphism $v\colon O_{X,x} \to k[\epsilon]/\epsilon^2$ to a solution, look at the images of x_i. To check that the solution is indeed based at $(a_1, \dots, a_n)$, note that if $v(x_i) = a_i' + \epsilon b_i$, then $v(x_i - a_i')$ is nilpotent, hence not a unit, but if $a_i' \neq a_i$ then $x_i - a_i'$ is a unit in $O_{X,x}$.

To go from a solution to a homomorphism, send x_i to $a_i + \epsilon b_i$ and then check that this extends to a homomorphism on all of $O_{X,x}$. You will have to divide, but division is easy in $k[\epsilon]/\epsilon^2$—anything with a non-zero constant term is invertible.

(2) — Complete the sketch above.

In the proof of 1.2.2, we saw that the "constant term" of $v(f)$ must be $f(x)$, that is v must have the form

$$v(f) = f(x) + \epsilon \cdot \delta(f)$$

where $\delta\colon O_{X,x} \to k$ is some function. Since v is a ring homomorphism, it satisfies

$$v(f+g) = v(f) + v(g) \text{ and } v(fg) = v(f)v(g).$$

In terms of δ, these become

$$\delta(f+g) = \delta(f) + \delta(g) \text{ and } \delta(fg) = f(x)\delta(g) + g(x)\delta(f). \tag{2}$$

Furthermore, for a constant function c, we have $v(c) = c$, and hence

$$\delta(c) = 0. \tag{3}$$

1.2.3 Derivations Equation (2) should remind you of the sum and product rule for derivatives. Maps $\delta\colon O_{X,x} \to k$ satisfying these equation are called *derivations*. If they also satisfy equation (3), then they are called k-*derivations* or derivations *over* k. This indicates that the elements of k in $O_{X,x}$ are to be treated as "constants". Denote by $\mathrm{Der}_k(O_{X,x})$ the set of k-derivations of $O_{X,x}$. Note that derivations can be added and multiplied by scalars (elements of k), which makes $\mathrm{Der}_k(O_{X,x})$ a k-vector space.

We saw that a k-algebra homomorphism $v\colon O_{X,x} \to k$ gives a k-derivation $\delta\colon O_{X,x} \to k$. Conversely, it is easy to check that a k-derivation $\delta\colon O_{X,x} \to k$ gives a k-algebra homomorphism $v(f) = f(x) + \epsilon \cdot \delta(f)$. Thus, a *tangent vector* to X at x is equivalent to a k-derivation of $O_{X,x}$.

Geometrically, the correspondance between curves and derivations is as follows. A curve in a space gives a recipe to differentiate a function; this is the directional derivative of the function in the direction of the curve. But to define the directional derivative, we don't need an actual curve, an "infinitesimal curve" will do. There is no way (that I know of) to make this precise in (differential) geometry, but it can be made perfectly precise in algebraic geometry using the ring $k[\epsilon]/\epsilon^2$.

1.2.4 Zariski tangent space Let $m \subset O_{X,x}$ be the maximal ideal. A derivation $\delta\colon O_{X,x} \to k$ restricted to m gives a k-linear map

$$\delta\colon m \to k$$

that takes m^2 to 0, and hence gives a map

$$\overline{\delta}\colon m/m^2 \to k.$$

Conversely, any k-linear map $w\colon m/m^2 \to k$ gives a derivation $\delta\colon O_{X,x} \to k$ defined by

$$\delta(f) = w(f - f(x)),$$

where $f(x)$ denotes the constant function on X with value $f(x)$. Thus, we get an isomorphism of vector spaces

$$\mathrm{Der}_k(O_{X,x}) \cong \mathrm{Hom}(m/m^2, k).$$

The space $\mathrm{Hom}(m/m^2, k)$ is called the *Zariski tangent space* and m/m^2 is called the *Zariski cotangent space* to X at x.

1.2.5 Computing the Zariski (co)tangent space Let $X \subset \mathbb{A}^n$ be affine with $I(X) = \langle f_1, \dots, f_r\rangle$ and let $x = (a_1, \dots, a_n)$ be a point of X. We know that $O_{X,x}$ is the quotient of $O_{\mathbb{A}^n,x}$ by $\langle f_1, \dots, f_r\rangle$. Let us denote the maximal ideal of $O_{\mathbb{A}^n,x}$ by $\mathfrak{m}$. Then $\mathfrak{m}$ is generated by $\langle x_1 - a_1, \dots, x_n - a_n\rangle$ and its square $\mathfrak{m}^2$ is generated by the pairwise products. As a result, $\mathfrak{m}/\mathfrak{m}^2$ has the k-basis $(x_1 - a_1, \dots, x_n - a_n)$. To get m/m^2, we need to further quotient by the polynomials $f_1, \dots, f_r$. Let $\overline{f}_1, \dots, \overline{f}_r$ denote the images of $f_1, \dots, f_r$ in $\mathfrak{m}/\mathfrak{m}^2$. Then

$$m/m^2 = \langle x_1 - a_1, \dots, x_n - a_n\rangle / \langle \overline{f}_1, \dots, \overline{f}_r\rangle.$$

But what *are* these mysterious $\overline{f}_1, \dots, \overline{f}_r$. They are not mysterious at all! We have

$$\overline{f}_i = \frac{\partial f_i}{\partial x_1}(a_1, \dots, a_n) \cdot (x_1 - a_1) + \dots + \frac{\partial f_i}{\partial x_n}(a_1, \dots, a_n)(x_n - a_n).$$

(3) — Prove the assertion above.

1.2.6 Examples (Hypersurfaces)

(4) — Compute the dimension of the tangent space of (a) $V(xy - z^2) \subset \mathbb{A}^3$ at $(0,0,0)$, (b) $V(XY - Z^2) \subset \mathbb{P}^2$ at $[0:1:0]$.

Let T_xX denote the tangent space of X at x.

1.2.7 Proposition (Dimension of the tangent space) We have $\dim T_xX \geq \dim_x X$.

Proof. (Sketch) I will give a proof using a result in commutative algebra called *Nakayama's lemma* and a fact about local rings. Neither of them are difficult once you develop the theory, but (again) their proper place is a course in commutative algebra.

Nakayama's lemma says the following: let R be a Noetherian local ring with maximal ideal m and let M be a finitely generated R-module. Consider $m_1, \dots, m_n \in M$ and their images $\overline{m}_1, \dots, \overline{m}_n$ in the R/m-vector space $\overline{M} = M/mM$. If $\overline{m}_1, \dots, \overline{m}_n$ span $\overline{M}$ as a vector space, then $m_1, \dots, m_n$ generate M as an R-module.

Let us apply it to $R = O_{X,x}$; its maximal ideal consists of the germs that vanish at x. It turns out that R is Noetherian. We take $M = m$ itself. Let $n = \dim m/m^2$ and let $\overline{m}_1, \dots, \overline{m}_n \in m$ be such that their images in m/m^2 form a basis. Then, by Nakayama's lemma, $m_1, \dots, m_n$ generate the ideal m.

We now "spread out" our knowledge from the germs $O_{X,x}$ to a Zariski neighborhood of x. Let $U \subset X$ be a small enough affine neighborhood of x such that $m_1, \dots, m_n$ are represented by functions on U. The maximal ideal of $O_{X,x}$ is the set of germs vanishing at x and we know that $m_1, \dots, m_n$ generate this ideal. If U is small enough, we can show that the functions $m_1, \dots, m_n$ generate the (maximal) ideal of $k[U]$ consisting of functions vanishing at x. As a result, the zero locus of the n regular functions $m_1, \dots, m_n$ on U is the point x. Using slicing dimension, we conclude that $n \geq \dim_x X$, which is what we set out to prove. $\square$

1.2.8 Definition (Non-singularity) We say that X is *smooth* or *non-singular* at x if

$$\dim_x X = \dim T_xX.$$

1.2.9 Examples Affine spaces, projective spaces, and Grassmannians are smooth at all points. So are their open subsets.

1.2.10 Examples (Hypersurfaces) $X = V(f) \subset \mathbb{A}^n$ is smooth at x if and only if at least one of the partial derivatives of f is non-zero at x.

(5) — Prove this.

1.2.11 Examples (Hypersurface) The Fermat cubic $V(X^3 + Y^3 + Z^3) \subset \mathbb{P}^2$ is smooth at every point on it.

(6) — Prove this.