1 Completeness of projective varieties

I have repeatedly asserted that projective varieties are the algebro-geometric analogue of compact topological spaces. In one sense, this is evident: over $\mathbb{C}$, the projective varieties are compact in the Euclidean topology. But we can abstract out a nice property of compact topological spaces and show that projective varieties satisfy this property (over any field).

1.1 Completeness

Recall that a continuous map of topological spaces $f\colon X \to Y$ is *closed* if it maps closed sets to closed sets. Not all continuous maps are closed; take for example, the map $f\colon \mathbb{A}^2 \to \mathbb{A}^1$ defined by $f(x, y) = x$. It sends the closed set $V(xy - 1)$ to the non-closed set $\mathbb{A}^1 \setminus \{0\}$.

1.1.1 Definition (Complete variety) We say that a variety X is *complete* if for any Y, the projection map

$$\pi\colon X \times Y \to Y$$

is closed.

1.2 Proposition (Closed image)

Let X be a complete variety, Y be a separated variety, and $f\colon X \to Y$ a regular map. Then the image $f(X)$ is closed in Y.

Proof. Consider the graph $\Gamma_f = \{(x, f(x)) \mid x \in X\} \subset X \times Y$. Note that this is the pre-image of the diagonal $\Delta \subset Y \times Y$ under the map $(f, id)\colon X \times Y \to Y \times Y$. Since Y is separated, Γ_f is closed. Since X is complete, the projection of Γ_f to Y is closed. But this projection is just the image of f. □

1.3 Theorem (Projective varieties are complete)

Let X be a projective variety. Then X is complete. That is, for any Y, the projection map $\pi\colon X \times Y \to Y$ is closed.

1.3.1 Remark Why is this a big deal? Let us consider an example, one we have seen in the homework. Let V be the vector space of homogeneous polynomials of degree d in X_0, X_1, X_2 and let $\Delta \subset V$ be the set of polynomials F that have a singularity at some point $p \in \mathbb{P}^2$. (This means that all three partials of F vanish at p). That is,

$$\Delta = \{F \mid \exists p \text{ such that } \frac{\partial F}{\partial X_i}(p) = 0 \text{ for } i = 0, 1, 2\}.$$

We want to prove that $\Delta \subset V$ is closed. Let us eliminate the existential quantifier by considering the set

$$Z = \{(F, p) \mid \frac{\partial F}{\partial X_i}(p) = 0 \text{ for } i = 0, 1, 2.\} \subset V \times \mathbb{P}^2.$$

It is easy to see that Z is closed: it is defined by polynomial equations in the coefficients of F and the coordinates of p. By definition, Δ is the image of Z under the projection map $V \times \mathbb{P}^2 \to V$. Since $\mathbb{P}^2$ is projective, hence complete, the image is closed.

The upshot is that Theorem 1.3 allows us to *eliminate existential quantifiers* as long as they are quantified over a complete variety. Note that the resulting statements about closedness can be extremely non-trivial. The fact that $\Delta \subset V$ is closed means that there is a system of polynomials in the coefficient of F that detects whether F has a singularity. (In the homework, you proved that Δ has codimension 1, which shows that the system consists of just one equation.)

1.3.2 Examples Here are some more examples of sets that we can show are closed by the same reasoning.

1. The subset of $\mathrm{Gr}(2,4) \times \mathrm{Gr}(2,4)$ consisting of (V, W) such that $V \cap W$ is non-zero.

2. Let $\mathbb{P}V$ be the projective space of surfaces of degree d in $\mathbb{P}^3$. The subset of $\mathbb{P}V$ consisting of surfaces that contain a line.

(1), (2) — Using Theorem 1.3, prove that the two sets mentioned above are closed.

1.3.3 Remark Intuitively, what does it mean that $\pi\colon X \times Y \to Y$ is closed? Suppose you have a family of points $(x_t, y_t) \in X \times Y$ such that $\lim_{t\to 0} y_t$ exists in Y. Then $\lim_{t\to 0} x_t$ must exist in X. That is, "points cannot escape to infinity in the X-direction."

We have the following very useful criterion for irreducibility in the context of closed maps.

1.4 Theorem (Closed maps and irreducibility)

Let $\pi\colon X \to Y$ be a surjective closed map of varieties such that Y is irreducibile and all fibers of π are irreducible of the same dimension. Then X is irreducible.

Proof. This is pure topology. Let n be the dimension of the fibers of π. Suppose $X = \bigcup X_i$ is the decomposition of X into irreducible components and let $\pi_i\colon X_i \to Y$ be the restriction of π. By the theorem on the dimension of fibers, there exists a non-empty open $U \subset Y$ such that $\dim \pi_i^{-1}(y)$ is constant as $y \in U$ (caution: it may be the case that $\pi_i^{-1}(y)$ is empty for some i; let us say that the empty set has dimension -1.) Let $n_i = \dim \pi_i^{-1}(y)$ for $y \in U$. Now, for some $y \in U$, we know that $\pi^{-1}(y) = \bigcup_i \pi_i^{-1}(y)$ has dimension n, so we must have $n = n_i$ for some i, say for $i = 1$. Since π is closed and $\pi(X_1)$ contains U, we must hae $\pi(X_1) = Y$. Thus by the theorem on the dimension of fibers, $\pi_1^{-1}(y)$ is itself non-empty of dimension at least n for every $y \in Y$. But we know that $\pi^{-1}(y) = \bigcup_i \pi_i^{-1}(y)$ is irreducible of dimension n. It follows that $\pi_i^{-1}(y) \subset \pi_1^{-1}(y)$ for all i and hence $\pi^{-1}(y) = \pi_1^{-1}(y)$. Since this holds for all y, we conclude that $X = X_1$. That is, X is irreducible. □

1.4.1 Example

(3), (4) — Using Theorem 1.4, prove that the two sets in Examples 1.3.2 are irreducible.

1.5 Proof of Theorem 1.3

We begin with a series of reductions.

1. If $P \times Y \to Y$ is closed and $X \subset P$ is a closed subset, then $X \times Y \to Y$ is also closed. Therefore, it suffices to treat the case of $P = \mathbb{P}^n$.

2. The map $P \times Y \to Y$ is closed if and only if there is an open cover $\{U_i\}$ of Y such that $P \times U_i \to U_i$ is clossed for all i. Hence, by passing to an affine cover, it suffices to treat the case where Y is affine.

3. If $Y \subset A$ is closed then the map $P \times Y \to Y$ is closed if and only if $P \times A \to A$ is closed. Therefore, it suffices to treat the case where Y is an affine space.

By the three reductions above, we are reduced to proving that the map

$$\mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m$$

is closed. Let $\pi\colon \mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m$ be the projection onto the second factor and let $Z \subset \mathbb{P}^n \times \mathbb{A}^m$ be a closed set. We want to prove that $\pi(Z)$ is closed; we prove that its complement is open.

What does Z look like? Choose homogeneous coordinates $[X_0 : \cdots : X_n]$ on $\mathbb{P}^n$ and coordinates $t_1, \ldots, t_m$ on $\mathbb{A}^m$. Then a closed set such as Z is the zero locus of a system of equations

$$F_i(X_0, \ldots, X_n, t_1, \ldots, t_m) = 0, \text{ for } i = 1, \ldots, r.$$

where each F_i is homogeneous in the X-coordinates (but not necessary in the t) coordinates. The set $\pi(Z)$ is the set of $(t_1, \ldots, t_m)$ for which the system has a non-zero solution and its complement is the set for which it does not have a non-zero solution. We must prove that if it does not have a non-zero solution for a particular choice of $(t_1, \ldots, t_m) = (a_1, \ldots, a_m)$, then there is a Zariski open subset around $(a_1, \ldots, a_m)$ such that for any $(t_1, \ldots, t_m)$ in this open set, the system does not have a non-zero solution. It follows from the Nullstellensatz that if a system of polynomial equations in X_i's has no non-zero solution then the radical of the ideal generated by the polynomials must be the ideal $(X_0, \ldots, X_n)$. Thus, there exists a large enough N such that any monomial in X_i lies in the ideal of $k[X_0, \ldots, X_n]$ generated by $F_i(X_0, \ldots, X_n, a_1, \ldots, a_m)$. Let us prove that the same is true if we replace $(a_1, \ldots, a_m)$ by any point in an open neighborhood.

Let V_ℓ denote the vector space of homogeneous polynomials of degree ℓ in $X_0, \ldots, X_n$. This is a finite dimensional space. Suppose the X-degree of F_i is d_i. For any $t = (t_1, \ldots, t_m) \in \mathbb{A}^m$, consider the map

$$M_t : \bigoplus_{i=1}^{r} V_{N-d_i} \to V_N$$

defined by

$$(g_1, \ldots, g_r) \mapsto F_1(X_0, \ldots, X_n, t_1, \ldots, t_m) g_1 + \cdots + F_r(X_0, \ldots, X_n, t_1, \ldots, t_m) g_r.$$

The domain and codomain of M_t are finite dimensional k-vector spaces and hence, after choosing bases, we can represent M_t by a matrix. The entries of this matrix may depend on t but they are polynomial functions of t.

Let $\nu = \dim V_N$. We know that for $t = (a_1, \ldots, a_m)$, the matrix of M_t has rank ν, because the map M_t is surjective. Thus, some $\nu \times \nu$ minor of M_t is non-zero at $t = (a_1, \ldots, a_m)$. Let $U \subset \mathbb{A}^m$ be the open subset containing $(a_1, \ldots, a_m)$ where this minor is non-zero. Then for any $t \in U$, the matrix of M_t has rank ν, which means that M_t is surjective. But this means that the system of equations $F_i = 0$ has no non-zero solutions in $X_0, \ldots, X_n$ for any $t \in U$. The proof is now complete.

(5) — To understand the proof, consider $Z \subset \mathbb{P}^1 \times \mathbb{A}^2$ defined by the equations

$$X^2 - sY^2 = 0 \text{ and } sX + tY = 0.$$

Notice that the point $(s, t) = (0, 1)$ is not in the image, and go through the proof to produce an open subset around $(0, 1)$ whose points are not in the image.

1.6 Consequences

1.6.1 Theorem (No global functions) Let X be a connected projective variety. Then the only regular functions on X are the constant functions.

Proof. A regular function is a regular map $f\colon X \to \mathbb{A}^1$ and hence it gives a regular map $\overline{f}\colon X \to \mathbb{P}^1$. Since X is complete, the image of $\overline{f}$ is closed. But the only closed subsets of $\mathbb{P}^1$ are $\mathbb{P}^1$ and finite sets. By construction, the image of $\overline{f}$ misses the point at infinity $[1 : 0]$, so the image must be a finite set. But X is connected, so the image is also connected, and hence must be a single point. Then f is a constant function. □