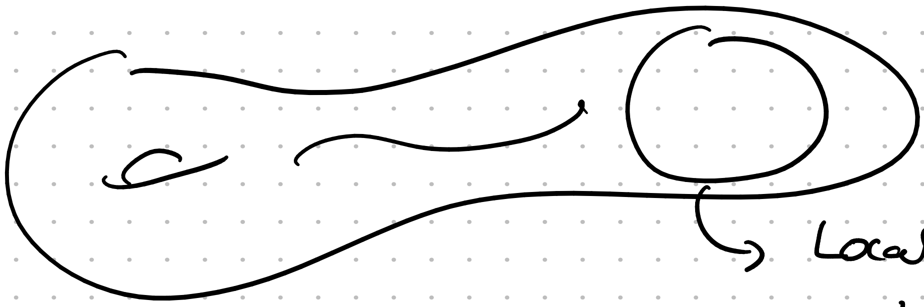
Towards: Algebraic Variety



Locally sol^{n}s of a system of eq^{n}s.

k a field.

$\mathbb{A}^n_k$ = Affine n-space over k
$= k^n = \{ (a_1, \dots, a_n) \mid a_i \in k \}$

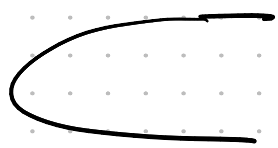
$\mathbb{A}^n$

$k[x_1, \dots, x_n]$ = Poly. ring over k in n vars.

$$\underset{\Psi}{f} : \mathbb{A}^n_k \longrightarrow k \quad \text{evaluation of } f.$$

$V(f) = \{ f(x_1, \dots, x_n) = 0 \} \subset \mathbb{A}^n_k$ "Vanishing set"

e.g. $\mathbb{A}^2_{\mathbb{R}} \supset V(y^2 - x)$



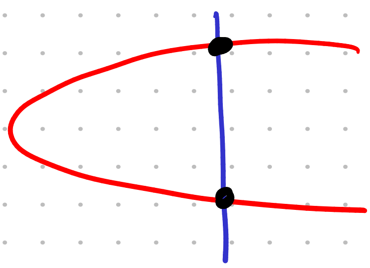
Generally, $A \subset k[x_1, \dots, x_n]$

$$V(A) = \{ (a_1, \dots, a_n) \mid f(a_1, \dots, a_n) = 0 \text{ for all } f \in A \}$$

$V(\phi) = \mathbb{A}^n$
$V(\{0\}) = \mathbb{A}^n$
$V(\{1\}) = \phi$

$\mathbb{A}^2$ $V(\{x, y\}) = \{(0,0)\}$

$V(\{y^2 - x, x - 5\})$



$$V(A) = \bigcap_{f \in A} V(f)$$

Given $A \subset k[x_1, \dots, x_n]$

Get $V(A) \subset \mathbb{A}^n_k$

These sets are called affine algebraic sets.) synonyms if k is alg. closed.
affine varieties.

$X \subset \mathbb{A}^n_k$ is an a.alg.set if $\exists\, A \subset k[x_1, \dots, x_n]$ such that $X = V(A)$.

1. $\emptyset$ is an affine var.
2. $\mathbb{A}^n_k$ is one.
3. Single points are.

Recall ideal of a ring R. ✓ Ring = associative commutative with 1

Given $A \subset R$

$$\langle A \rangle := \text{ideal gen. by } A$$
$$= \left\{ r_1 a_1 + \dots + r_m a_m \;\middle|\; \begin{array}{l} r_i \in R \\ a_i \in A \end{array} \right\}$$

Prop: $V(A) = V(\langle A \rangle)$

Pf: $V(\langle A \rangle) \subset V(A)$ because $A \subset \langle A \rangle$

Conversely if $(P_1, \dots, P_n) \in V(A)$

Need to show every $f \in \langle A \rangle$ vanishes on $P = (P_1, \dots, P_n)$

$$f = r_1 a_1 + \dots + r_m a_m$$
$$f(P) = r_1(P) \cdot \underbrace{a_1(P)}_{0} + \dots + r_m(P)\, \underbrace{a_m(P)}_{0}$$
$$= 0$$

$\square$

Affine alg. subsets of $\mathbb{A}^1_k$

$V(I)$ where $I \subset k[x]$ an ideal.

I must be principal

$I = \langle f \rangle$ $\qquad V(I) = V(f)$

- $f = 0$ then $\mathbb{A}^1_k$
- $f \neq 0$ then finite.

Hilbert Basis Thm

Thm: Every ideal I of $k[x_1, \dots, x_n]$ is finitely generated.

i.e. $I = \langle \{f_1, \dots, f_r\} \rangle$

$\Rightarrow$ A system of poly. eq^{n}s $\iff$ A finite system of eq^{n}s.

Thm: Every ideal of $k[x_1, \dots, x_n]$ is fin. gen.

Def: A Noetherian ring is a ring R whose every ideal if fin. gen.

$k[x_1, \dots, x_n]$ is Noetherian.

Pf idea - inductive.

R Noetherian $\Rightarrow$ $R[x]$ is also Noetherian.

Reminiscent of div algorithm in one var.

Be careful about leading coeff!