Regular functions

Affine variety = Zariski closed subset of $\mathbb{A}^n_k$

Quasi-affine = Zariski open of an affine

$$
\begin{array}{lcl}
\text{affine} \hookrightarrow & X & \\
& \text{Closed} \subset \mathbb{A}^n_k & \\
& \cup & \\
\text{quasi affine} \hookrightarrow & \text{Open subset of } X & \\
& \parallel & \\
& \underline{\underline{X}} \cap \underline{\underline{\text{Open}}} \text{ in } \mathbb{A}^n_k &
\end{array}
$$

Quasi-affine = Closed in $\mathbb{A}^n_k$ $\cap$ open in $\mathbb{A}^n_k$.

Building blocks

Ex.

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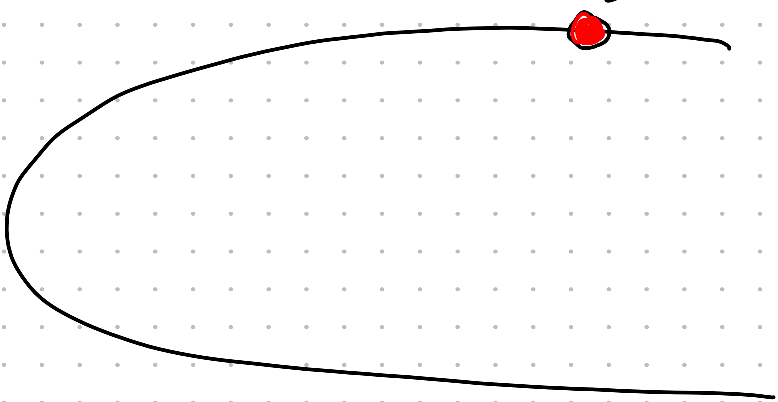


Def of reg. fun:

$S \subset \mathbb{A}^n_k$ $\qquad f: S \to k$

$a \in S$. We say f is "regular" at a if locally around a, we can express f as

$$f = \frac{p}{q} \qquad p, q \in k[X_1, \dots, X_n]$$

$q(a) \neq 0$.

→ There exists an open $U \subset S$ such that $a \in U$ &

$$f(x) = \frac{p(x)}{q(x)} \quad \forall \ x \in U$$

"Regular on S" = Regular at all pts of S.

Examples. ① $S = \mathbb{A}^1 - \{0\} \subset \mathbb{A}^1_t$

$f(t) = \frac{1}{t}$ is regular on S.

② Every constant function.

③ Every polynomial in $k[x_1, \dots, x_n]$ defines a rg. fun on $S \subset \mathbb{A}^n_k$.

④ $\underline{S} = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \text{ of } \underline{\text{rank} \leq 1} \quad \underline{[z,w] \neq [0,0]} \right\}$

$\subset \mathbb{A}^4$

$V(xw - yz) \leftarrow$ closed

$\{[z,w] \neq [0,0] \text{ is open}\} \leftarrow$ open.

$= \{z \neq 0\} \cup \{w \neq 0\}$

$f : S \to k$ is "either $\frac{x}{z}$ or $\frac{y}{w}$"

is $\underline{\text{regular}}$.

Properties: ① Constants are regular
② Sum, products of reg. fun are regular.

Def: $k[S] = $ The ring of reg. fun on S.
$\overset{\shortparallel}{\mathcal{O}(S)}$ ↪ k-algebra.

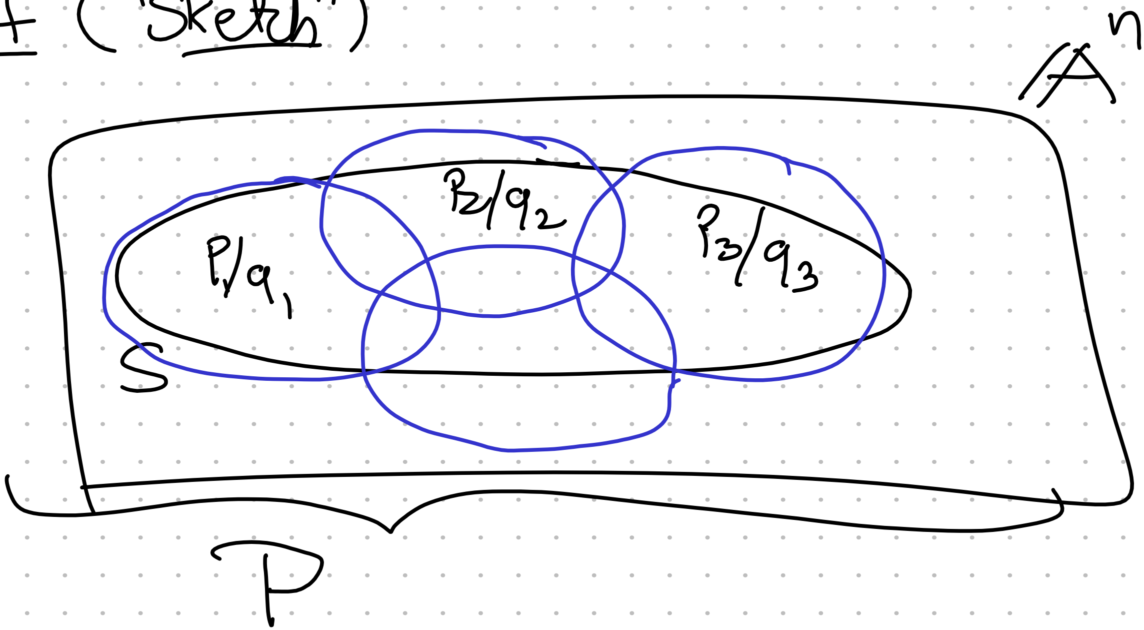
We have an obv map $k \to k[S]$
$c \mapsto$ const. fun c

$k[S]$ is nilpotent free.

k alg. closed.

Thm: $S \subset \mathbb{A}^n_k$ Zariski closed. Then every $f \in k[S]$ is defined by a polynomial in $k[x_1, \dots, x_n]$.

Pf ("Sketch")



$\square$

Cor: $$k[x_1, \ldots, x_n] \twoheadrightarrow k[S]$$

is surjective if $S \subset \mathbb{A}^n$ is closed.

$$\ker = I(S)$$

So we have an iso

$$\boxed{k[x_1, \ldots, x_n]/I(S) \xrightarrow{\sim} k[S].}$$

Regular maps

$$\begin{array}{ccc} \varphi : S & \longrightarrow & T \\ \cap & & \cap \\ \mathbb{A}^n & & \mathbb{A}^m \end{array}$$ is regular if

$$\varphi(s) = (\underbrace{\varphi_1(s)}, \ldots, \underbrace{\varphi_m(s)})$$

all regular.

Defines a category $\underline{\underline{QAff}}_k$

Obj are quasi-affine variety
morphism are regular maps.

<u>Ex</u>: ① $\varphi : S \to \mathbb{A}^1$ regular map

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\parallel$

$\qquad\qquad\qquad\qquad\qquad\qquad$ regular function.

② $S = \mathbb{A}^1 \setminus \{0\}$

$T = V(xy-1) \subset \mathbb{A}^2$

$\{(x,y) \mid xy = 1\}$

$\varphi : S \to T \qquad \varphi(t) = \left(t, \frac{1}{t}\right)$

$\psi : T \to S \qquad \psi(x,y) = x$

$\varphi \circ \psi = id \qquad \psi \circ \varphi = id.$

y

① Id is regular.

② Composition of rg. maps is regular.

$$\frac{x+1}{y} \qquad x = \frac{u}{v} \qquad y = \frac{u^2}{v^3}$$

$\rightsquigarrow$ fraction in u, v.

substituting fractions in fractions produces fractions.

$\| \varphi : S \to T$ regular map

$f : T \to k$ reg. function.

Get $f \circ \varphi : \underline{\underline{S}} \to k$ is regular.

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[dr, "f\circ\varphi"'] & T \arrow[d, "f"] \\ & \mathbb{A}^1 \end{tikzcd}$$

$\| \circ\varphi : k[T] \xrightarrow{\varphi^*} k[S].$

k-alg. hom.

<u>Contr.</u> f<u>unctor</u>

$$\mathrm{Quaff}_k \longrightarrow k\text{-alg.}$$

$$S \longmapsto k[S]$$

$$S \xrightarrow{\varphi} T \longmapsto \varphi^* : k[T] \to k[S]$$

Thm: If $\underset{\mathbb{A}^n}{\underset{\cap}{S}}, \underset{\mathbb{A}^m}{\underset{\cap}{T}}$ are closed

then for every

$$\psi : k[T] \to k[S]$$

$\exists!$ $\varphi : S \to T$ such that

$$\psi = \varphi^*.$$

In general

$\varphi : S \longrightarrow T$ reg. map

for affines

$\varphi^* : k[T] \longrightarrow k[S]$ hom of k-alg.