ALGEBRAIC VARIETIES

From quasi affine to general alg. varieties.

(Analogous to going from open subsets of $\mathbb{R}^n$ to general manifolds.)

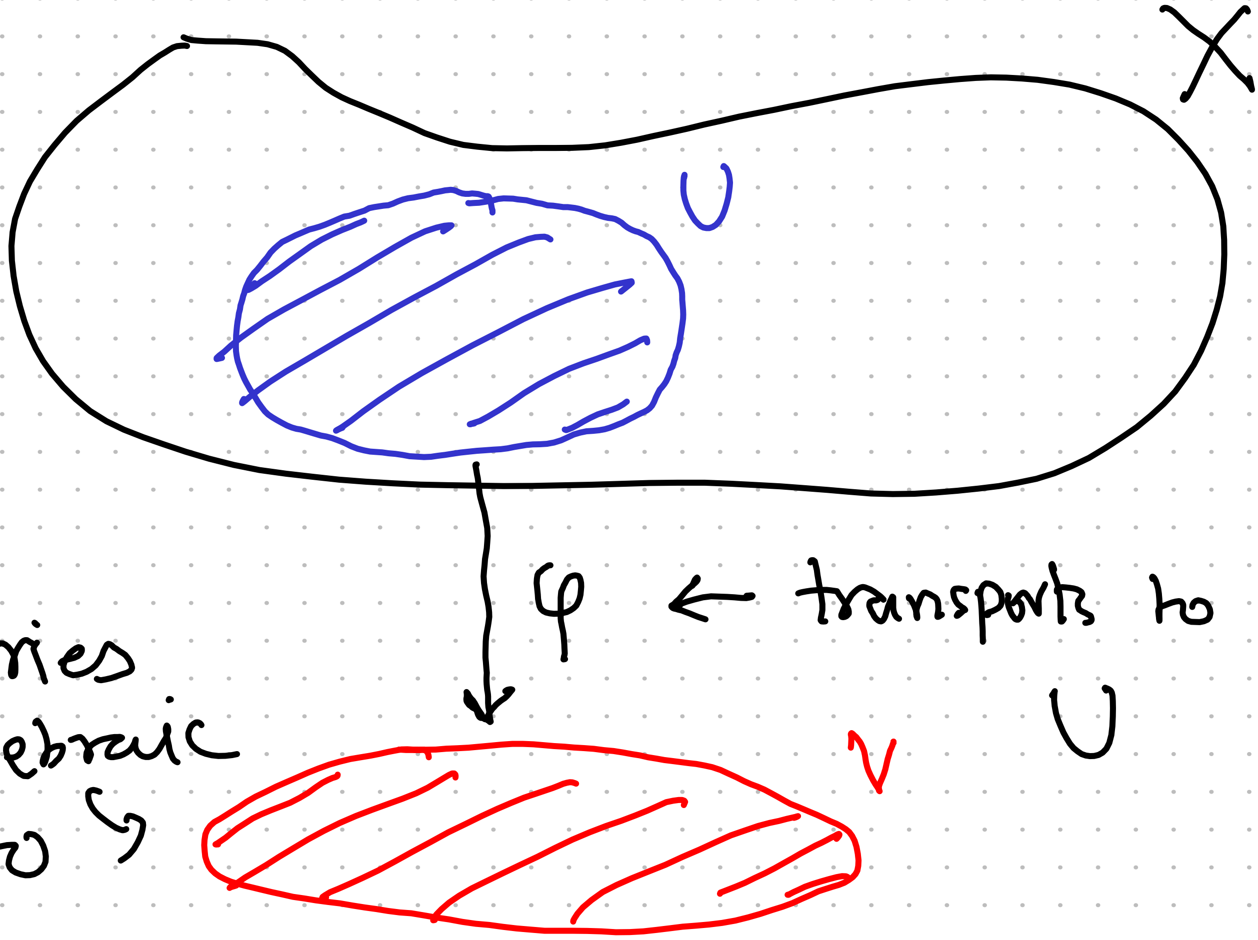
Let X be a top. space

An algebraic chart on X is

① $U \subset X$ open

② $\phi : U \to V$ a homeo where V is a quasi-affine.

Carries algebraic into



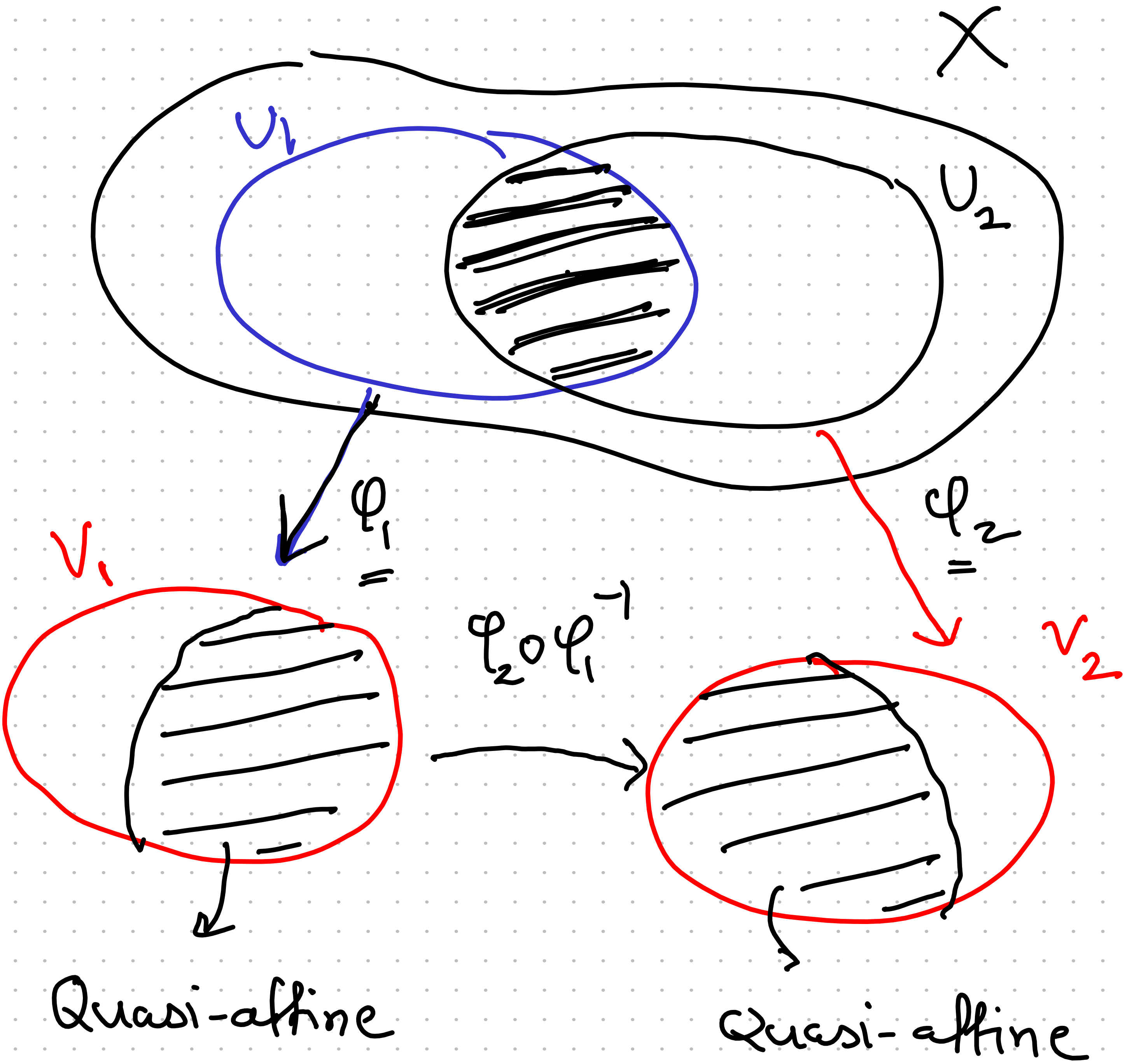
$\leftarrow$ transports to U

Compatible Charts

$$\phi_1 : U_1 \to V_1$$

$$\phi_2 : U_2 \to V_2 .$$

Two charts



$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is a homeo.

Charts are compatible if $\varphi_2 \circ \varphi_1^{-1}$ is a bi-regular isomorphism.

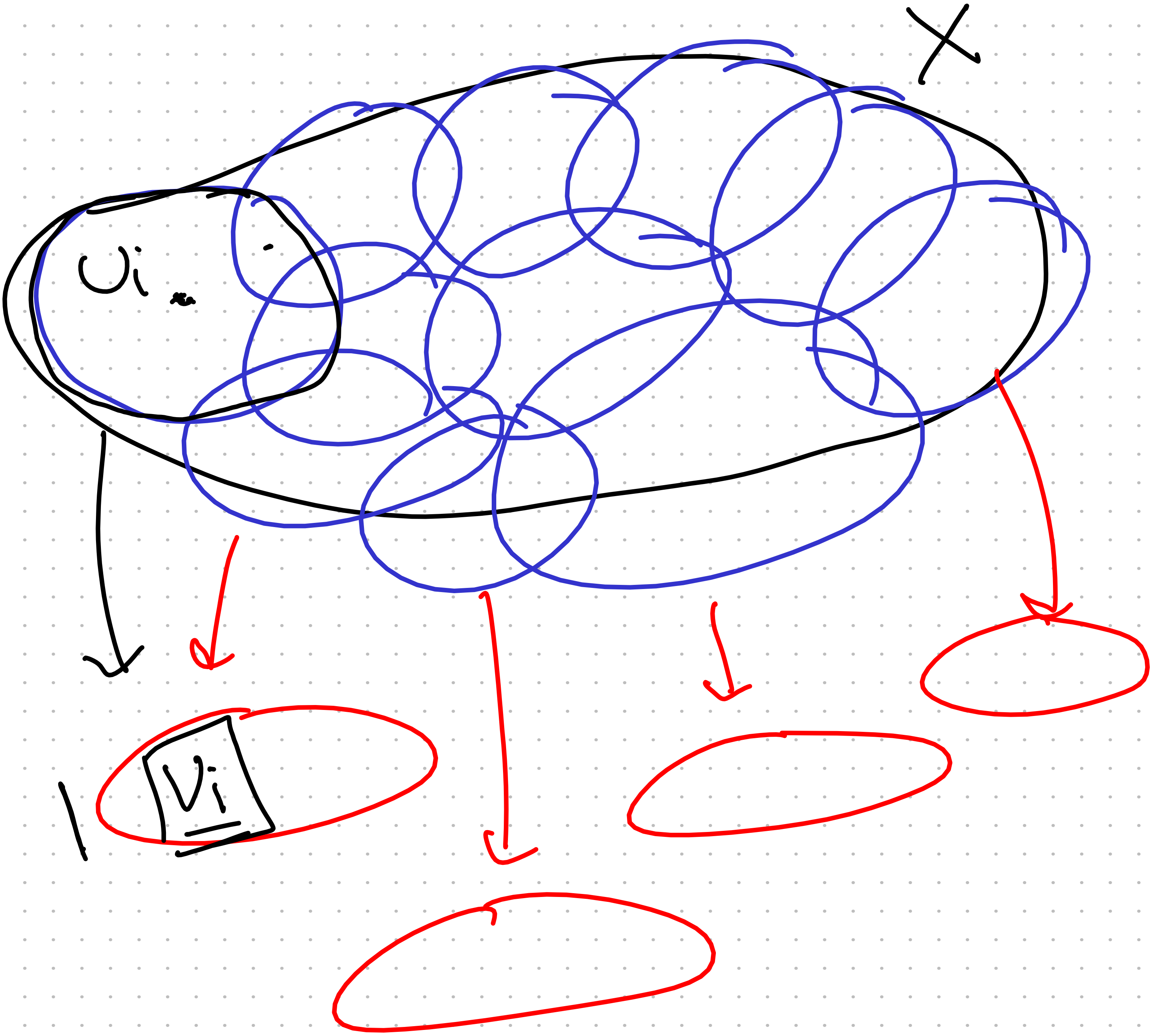
Definition: An algebraic variety is a topological space X together with an algebraic atlas.

Atlas :- A collection of charts $\{\phi_i : U_i \to V_i\}$ such that

ⓐ $\{U_i\}$ cover X, i.e.

$$\bigcup U_i = X$$

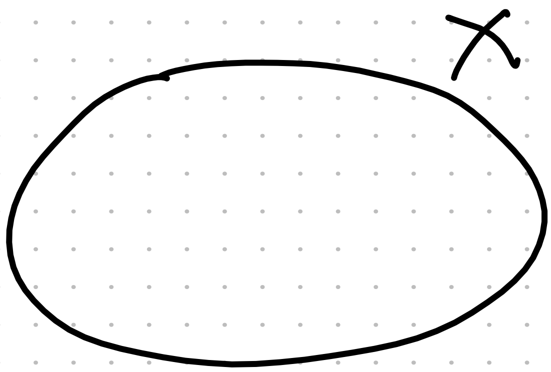
ⓑ Pairwise compatible.



Every point has an "algebraic reference frame" around it
↳ **chart**.

Examples

① Any quasi-affine variety is naturally an algebraic variety.
↳ obvious atlas.



$U = X$

$\{ id : X \to X \}$

② Lots more.

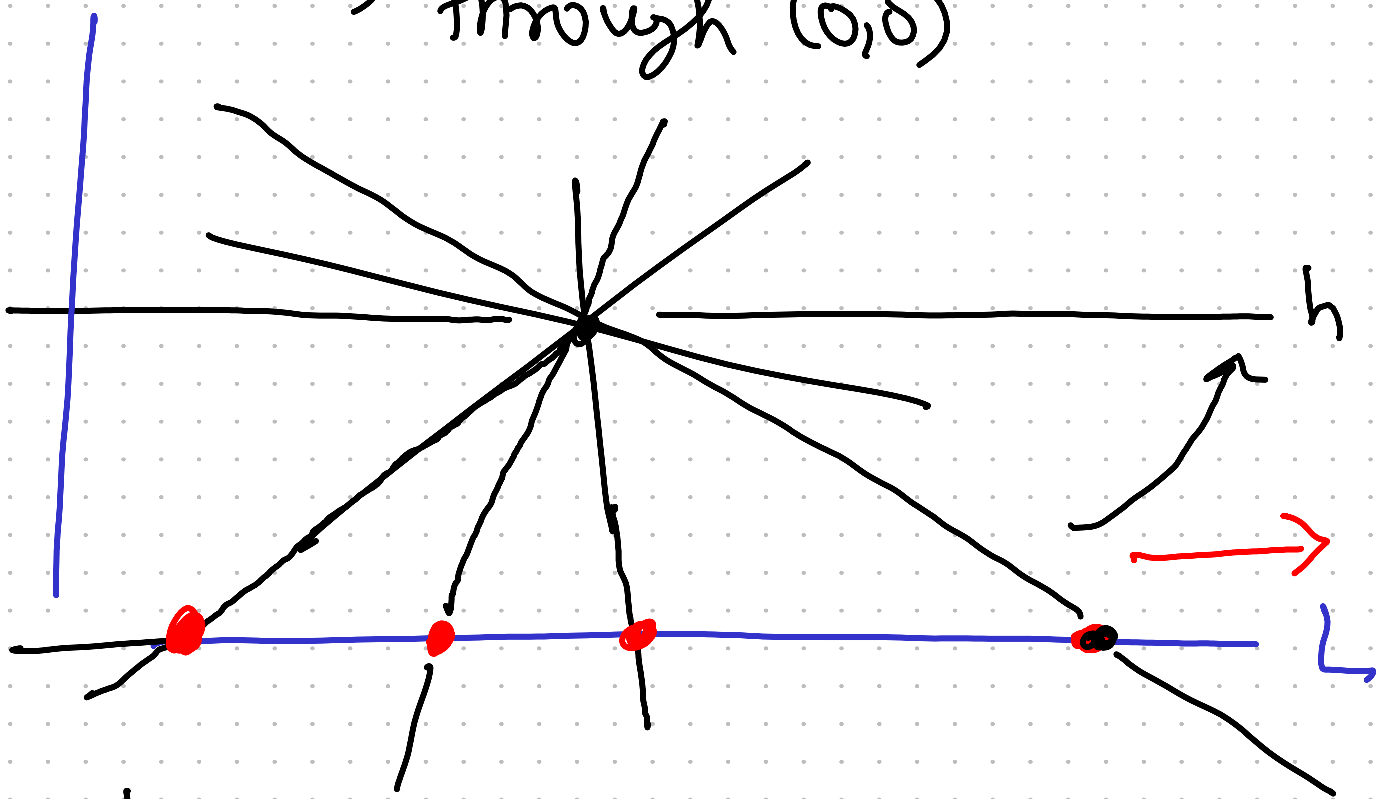
Projective Space $\mathbb{P}^n_k$

$\mathbb{P}^n_k$ = Set of $\underbrace{\text{1-dim subspaces}}_{\text{"line"}}$ of k^{n+1}!

Pictures: $(k = \mathbb{R})$

$\mathbb{P}^1$ = Lines in $\mathbb{R}^2$

through $(0,0)$

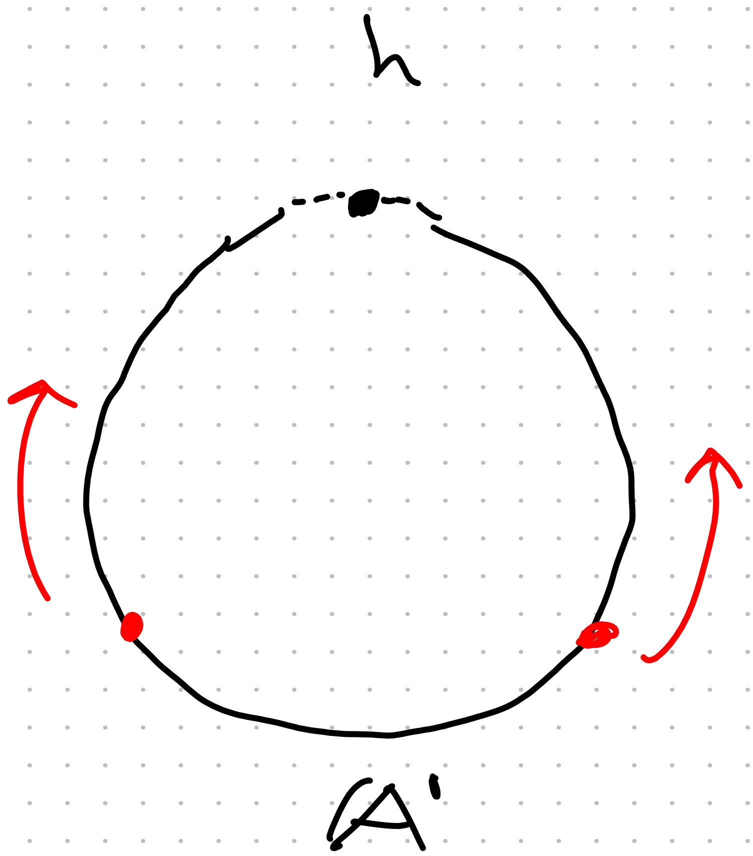


$\mathbb{P}^1 \setminus \{h\} \longleftrightarrow L \cong \mathbb{A}^1$

$\mathbb{P}^1 = \mathbb{A}^1 \cup \{\text{extra point}\}$

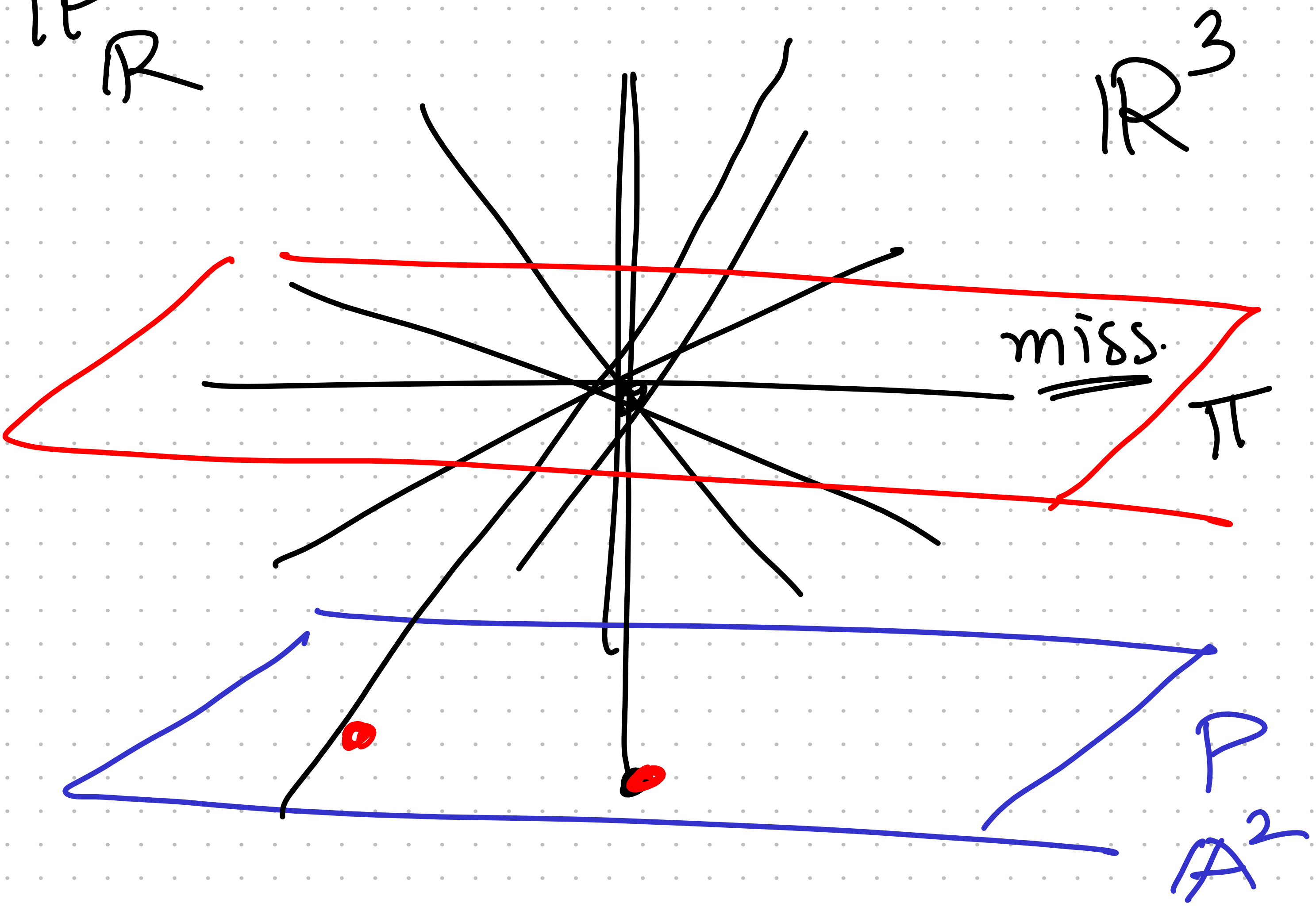
<u>NOT</u>:

<u>Correct</u>:



In Euclidean top, $\mathbb{A}^1_{\mathbb{C}/\mathbb{R}}$ not compact, but $\mathbb{P}^1$ is & it is the one pt. compactification.

$\mathbb{P}^2_{\mathbb{R}}$



$$\mathbb{P}^2 = \mathbb{A}^2 \cup \{\underline{\text{Lines in } \Pi}\}$$
$$= \mathbb{A}^2 \cup \mathbb{P}^1$$

$$\mathbb{P}^n = \underline{\underline{\mathbb{A}^n}} \cup \underline{\underline{\mathbb{P}^{n-1}}}$$

Not
canonical

Zariski Topology on $\mathbb{P}^n$ & Charts

$\mathbb{P}^n = \{$1-dim subs of $k^{n+1}\}$

↓

Spanned by $(a_0, \dots, a_n)$

$\in k^{n+1} \setminus \{0\}$

$(a_0, \dots, a_n)$ & $(b_0, \dots, b_n)$ have same span iff

$$(b_0, \dots, b_n) = \lambda (a_0, \dots, a_n)$$

$$\lambda \in k^\times$$

$$\mathbb{P}^n = \{(a_0, \dots, a_n) \in k^{n+1} \setminus 0\} / \sim$$

$(a_0, \dots, a_n) \sim (b_0, \dots, b_n)$ if

$\exists \lambda$ s.t. $b = \lambda \cdot a$.

$\underline{\underline{\mathbb{P}^n}} = (\underline{\underline{\mathbb{A}^{n+1} \setminus \{0\}}}) / \text{scaling}.$

Topology on $\mathbb{P}^n$ is the quotient topology.

<u>Reminder</u>: $U \subset \mathbb{P}^n$ is open/cl iff its preimage in $\mathbb{A}^{n+1} \setminus \{0\}$ is open/cl.

<u>Notation</u>:

$$[a_0 : \cdots : a_n] = \text{Eqv. class of } (a_0, \ldots, a_n)$$

$$[a_0 : \cdots : a_n] = [\lambda a_0 : \cdots : \lambda a_n]$$

Consider

$\underline{U_0} \subset \mathbb{P}^n$ is open

$= \{ [a_0 : \cdots : a_n] \mid a_0 \neq 0 \}$

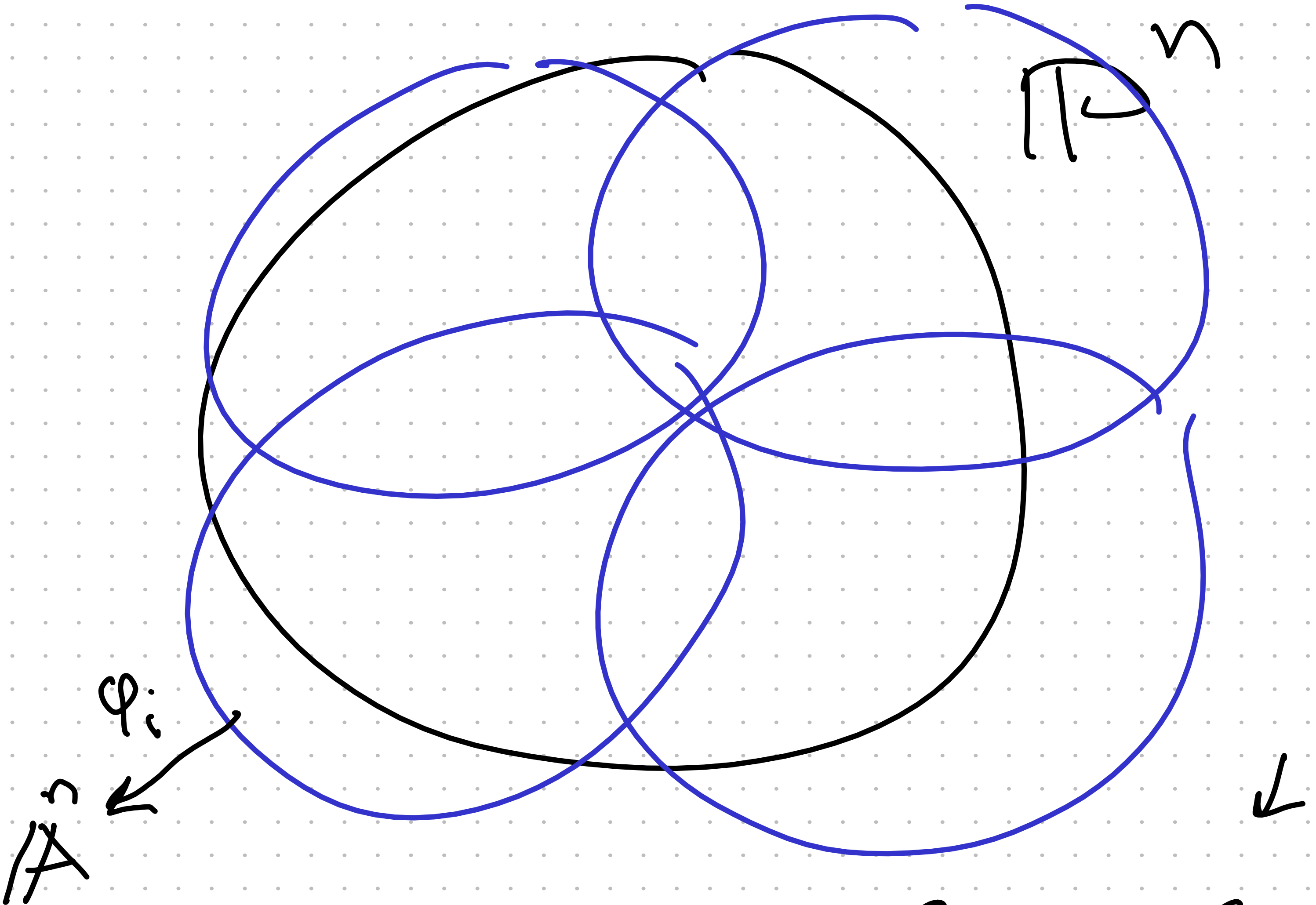
Preimage in $\mathbb{A}^{n+1} \setminus \{0\}$ is open.

$= \{ (a_0, \ldots, a_n) \mid a_0 \neq 0 \}$

$U_1, U_2, \ldots, U_n$

$U_i = \{ [a_0 : \cdots : a_n] \mid a_i \neq 0 \}$

$\mathbb{P}^n = \bigcup U_i$.



$$[a_0 : \cdots : a_n] = [\underline{1} : \frac{a_1}{a_0} : \cdots : \frac{a_n}{a_0}]$$

$$U_0 = \{ [1 : \underbrace{\alpha_1 : \cdots : \alpha_n}] \mid \alpha_i \in k \}$$

$$\downarrow (\alpha_1, \dots, \alpha_n)$$ bijection.

$$\underline{\underline{\mathbb{A}^n}}$$

($U_0 \to \mathbb{A}^n$ is a homeo)

$\mathbb{P}^n$ "looks locally like $\mathbb{A}^n$ everywhere"

Every pt. has a neighborhood <u>isomorphic</u> to $\mathbb{A}^n$

(Classwork - check <u>compatibility</u>).

<u>Easy</u>: Open/Closed $\subset$ <u>Alg. Variety</u>

↳ Becomes an alg. variety by restricting the <u>Charts</u>.