Regular maps (for alg var)

Regular function:

X an algebraic variety

$f : X \to k$ a function.

continuous

$x \in X$ any point:

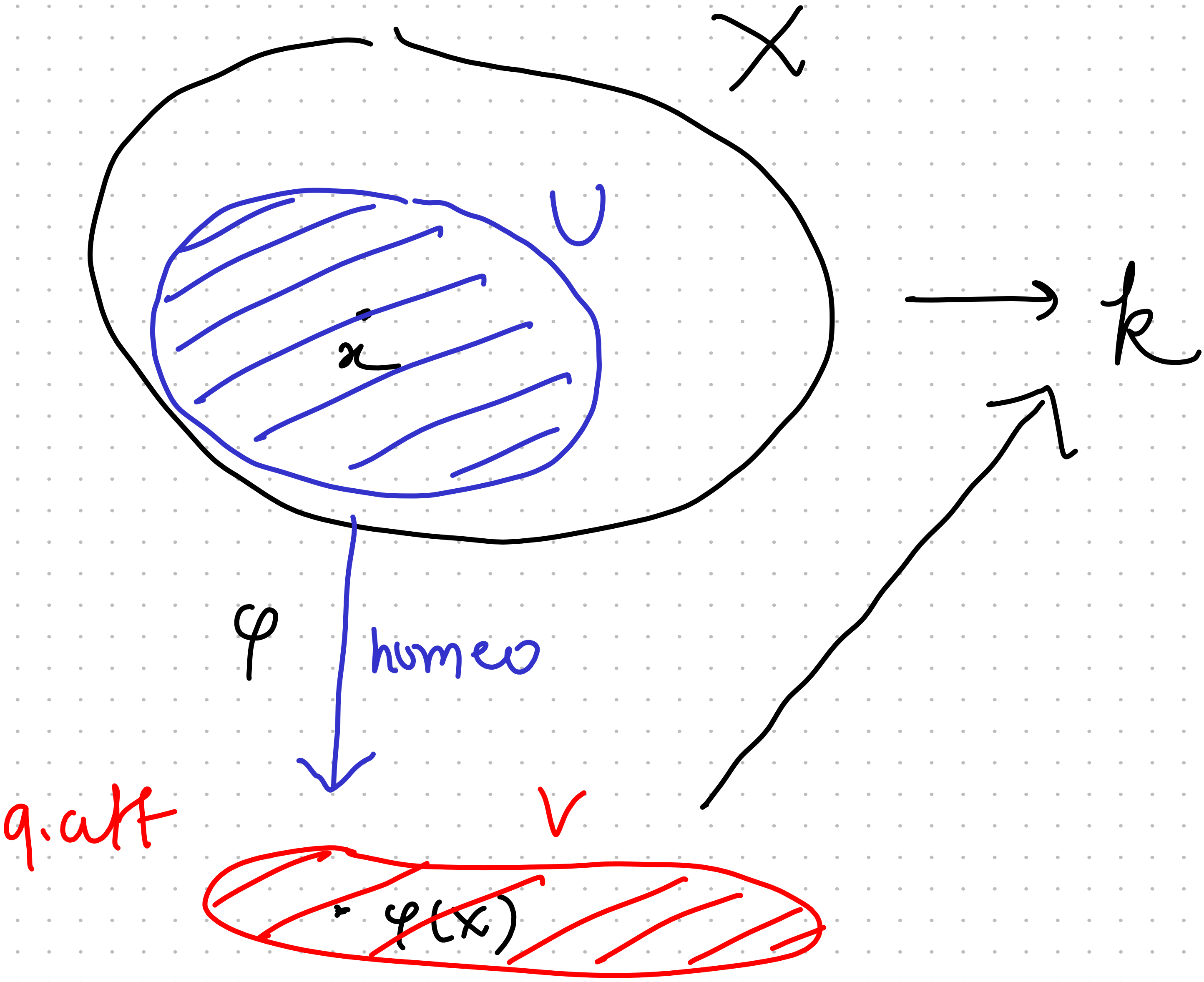
f is regular at x if

there is a chart $\varphi : U \to V$

with $x \in U$ such that

$$f \circ \varphi^{-1} : V \longrightarrow k$$

is regular at $\varphi(x)$.



Rmk: If $f \circ \varphi^{-1}$ is regular for one chart, then compatibility $\Rightarrow$ is regular for every chart.

So there is $\Leftrightarrow$ for every

Example: $X = \mathbb{P}^1$

$$= \{ [X:Y] \mid \text{not both zero} \}$$

$$f([X:Y]) = \frac{X}{Y}$$

a function on $\mathbb{P}^1 \setminus \{[1:0]\}$

is regular!

(check on charts)

$$\mathbb{P}^1 = U_0 \cup U_1$$

$$U_0 = \{[1:\underline{Y}]\} \qquad U_1 = \{[\underline{X}:1]\}$$

$$\varphi_0 : U_0 \to \mathbb{A}^1 \qquad \varphi_1 : U_1 \to \mathbb{A}^1$$

$$f \circ \varphi_0^{-1} : \mathbb{A}^1 \setminus 0 \to k \qquad f \circ \varphi_1^{-1} : \mathbb{A}^1 \to k$$

$$y \mapsto \frac{1}{y} \qquad\qquad x \mapsto x$$

Example: $X = \mathbb{P}^n$

F, G two homog. poly of the same degree.

$$f : X \setminus V(G) \longrightarrow k$$

$$[x_0 : \cdots : x_n] \longmapsto \frac{F(x_0, \ldots, x_n)}{G(x_0, \ldots, x_n)}$$

is regular.

Fact: The only regular funct. on $\mathbb{P}^n$ are the <u>constants</u>

$$k[\mathbb{P}^n] \cong k$$

Regular Maps

X, Y alg. var. $\qquad y = f(x)$

$f : X \longrightarrow Y$ continuous.

$x \in X$. f is regular at x

if for some (eqv. for every)

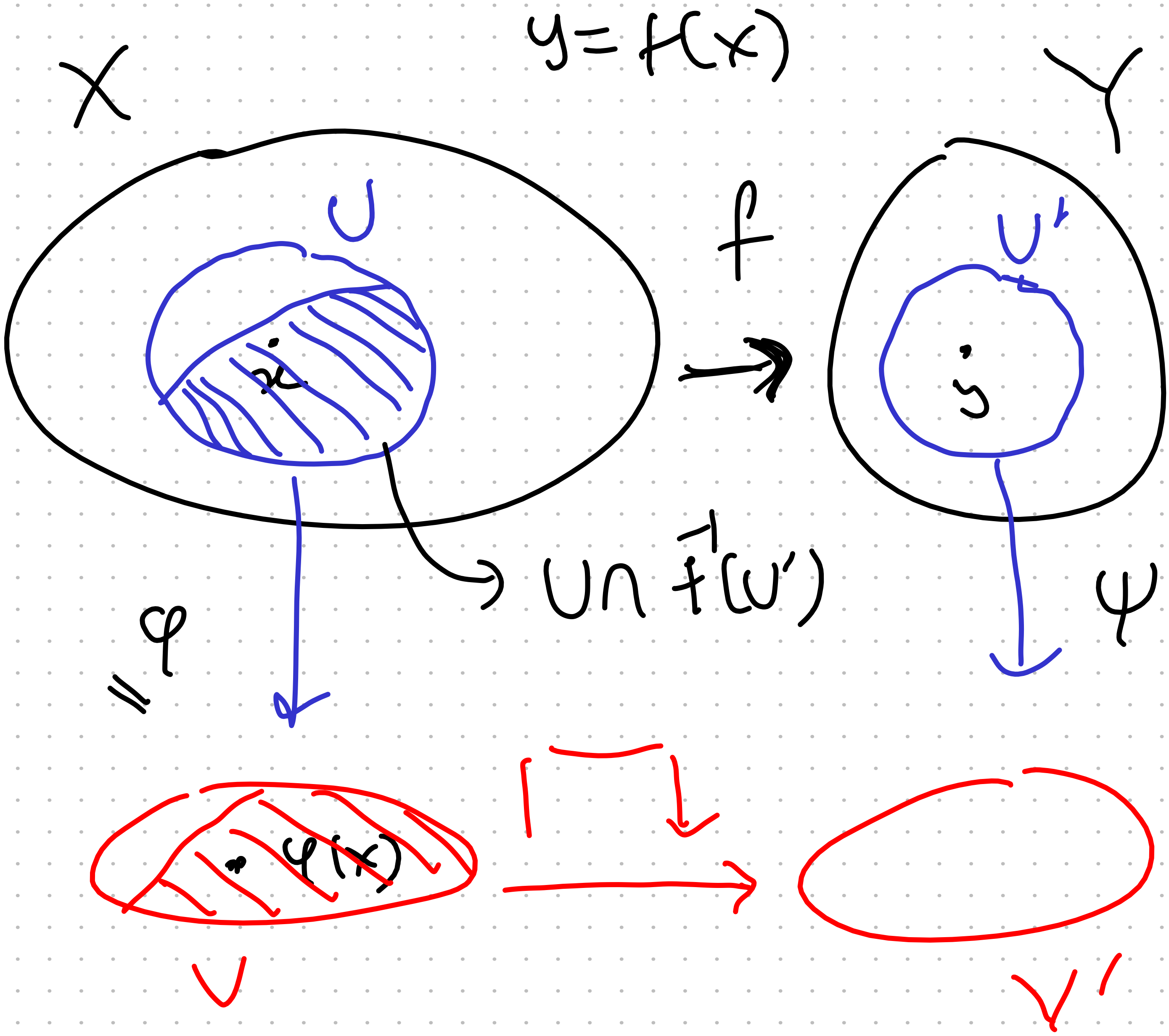
charts $\varphi : U \to V$ around x

& $\psi : U' \to V'$ around y

the function $\qquad$ fake domain

$\psi \circ f \circ \varphi^{-1} : V \dashrightarrow V'$

is reg. at $\varphi(x)$ $\qquad$ defined on an open



f is regular at x if

$\psi \circ f \circ \varphi^{-1}$ is regular at

$\varphi(x)$.

Example.

$$\mathbb{P}^2 \overset{f}{\dashrightarrow} \mathbb{P}^1 = [U:V]$$

$$[X:Y:Z] \mapsto [XZ : X^2+Y^2]$$

$$f^{-1}(V(F(U,V)))$$

$$= V(F(XZ, X^2+Y^2))$$

Domain of f includes all pts where at least one XZ, X^2+Y^2 is non-zero.

$$[1:0:0] \longmapsto [0:1]$$

$$\begin{array}{ccc} [1:0:0] & \longmapsto & [0:1] \\ \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ x & \longmapsto & y \end{array}$$

Charts.

$$\begin{tikzcd}
x \quad X \neq 0 \arrow[r] & \cdot\, y \quad V \neq 0 \\
{[1:y:z]} \arrow[d, "\varphi"', "{(y,z)}"] \arrow[r, bend left] & {[u:1]} \arrow[d, "u"] \\
\mathbb{A}^2 & \mathbb{A}^1
\end{tikzcd}$$

$$\begin{matrix} \mathbb{A}^2 \\ \ni \\ (0,0) \end{matrix}$$

$$\begin{tikzcd}
\underline{\underline{[1:y:z]}} \arrow[r, "f"] & {[z : 1+y^2]} \arrow[d] \\
{(y,z)} \arrow[u] \arrow[r, dashed, "f"] & \dfrac{z}{1+y^2}
\end{tikzcd}$$

$$\underline{\underline{(0,0)}}$$

$F_0, \ldots, F_m \in k[X_0, \ldots, X_n]$

homog of <u>same</u> degree.

Then we get

$$\mathbb{P}^n \dashrightarrow \mathbb{P}^m$$

$$[X_0 : \cdots : X_n] \to [F_0(X_0, \ldots, X_n) : F_1(X_0, \ldots, X_n) : \cdots : F_m(X_0, \ldots, X_n)]$$

<u>Domain</u> = Complement of $V(F_0, \ldots, F_m)$.

$$f: \mathbb{P}^1 \longrightarrow \mathbb{P}^2 = [U:V:W]$$

$$\underline{[X:Y]} \longrightarrow \underline{[\underset{=}{X^2} : \underset{=}{XY} : \underset{=}{Y^2}]}$$

$$V(X^2, Y^2, XY) = \emptyset$$

$$\text{Image}(f) \subseteq \underline{V(UW - V^2)}$$

$$g: V(\underline{UW - V^2}) \dashrightarrow \mathbb{P}^1$$

$$[U:V:W] \longmapsto [U:V]$$

undefined at $[0:0:1]$

$$g': V(UW - V^2) \dashrightarrow \mathbb{P}^1$$

$$[U:V:W] \longmapsto [V:W]$$

is defined at $[0:0:1]$ ✓

g & g' agree on the common domain. $\Downarrow$

$[U:V] = [V:W]$

on $V(UW - V^2)$

g extends to a reg. map

$V(UW - V^2) \longrightarrow \mathbb{P}^1$

is the inverse to f.

$$
\begin{tikzcd}
{[x:y]} \arrow[r, "f"] & {[x^2:xy:y^2]} \arrow[d, "g"] \arrow[dr] & \\
& {[x^2:xy]} \arrow[d, phantom, "\parallel"] & {[xy:y^2]} \arrow[d, phantom, "\parallel"] \\
& {[x:y]} & {[x:y]}
\end{tikzcd}
$$

$$f: \mathbb{P}^1 \longrightarrow \mathbb{P}^2$$

$$= \underline{[X^2 : XY : Y^2]}$$

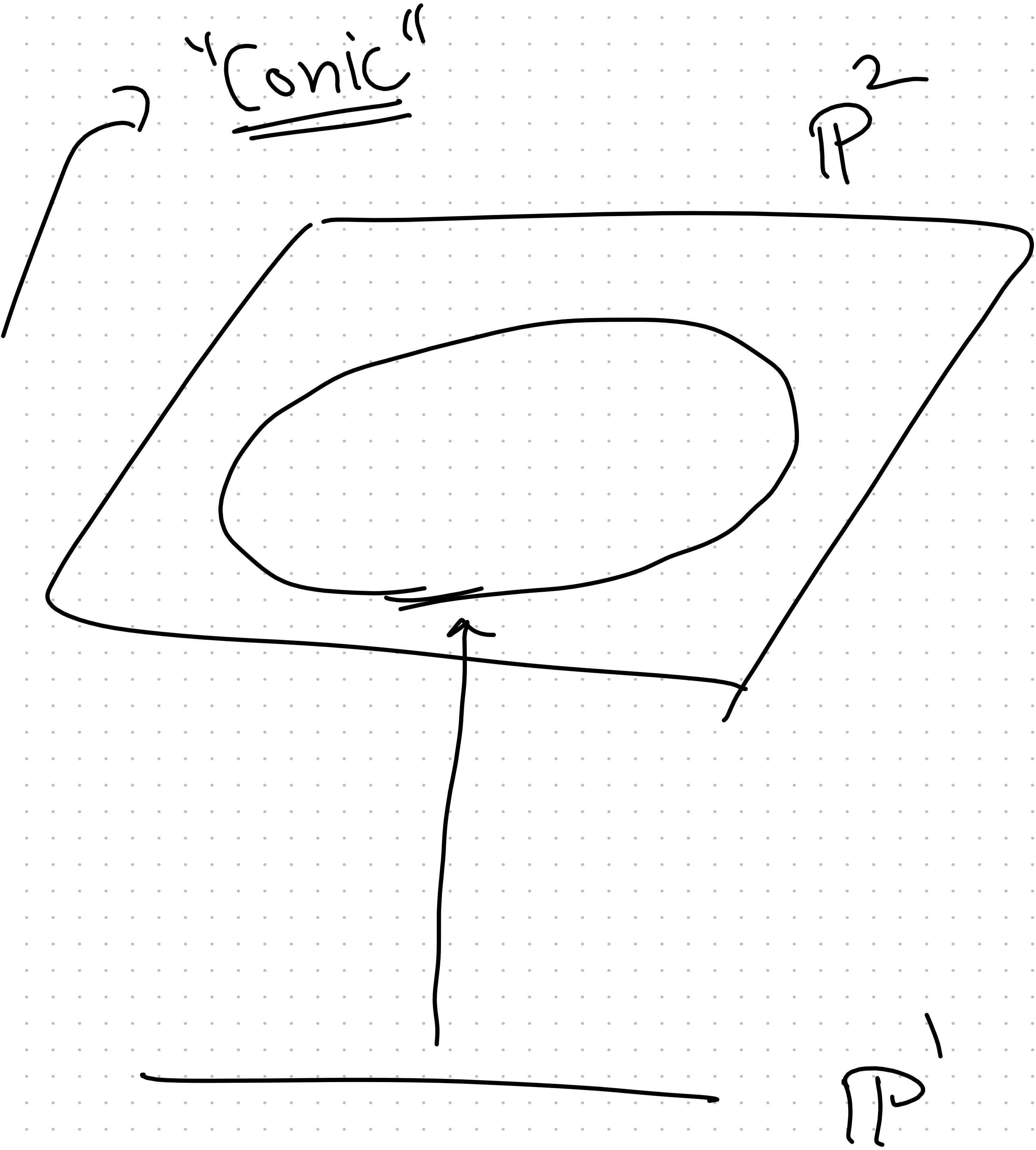
Im(f) is closed in $\mathbb{P}^2$

& f is an isomorphism onto its image.

$$\boxed{V(UV - W^2) \subset \mathbb{P}^2}$$

$$\cong$$

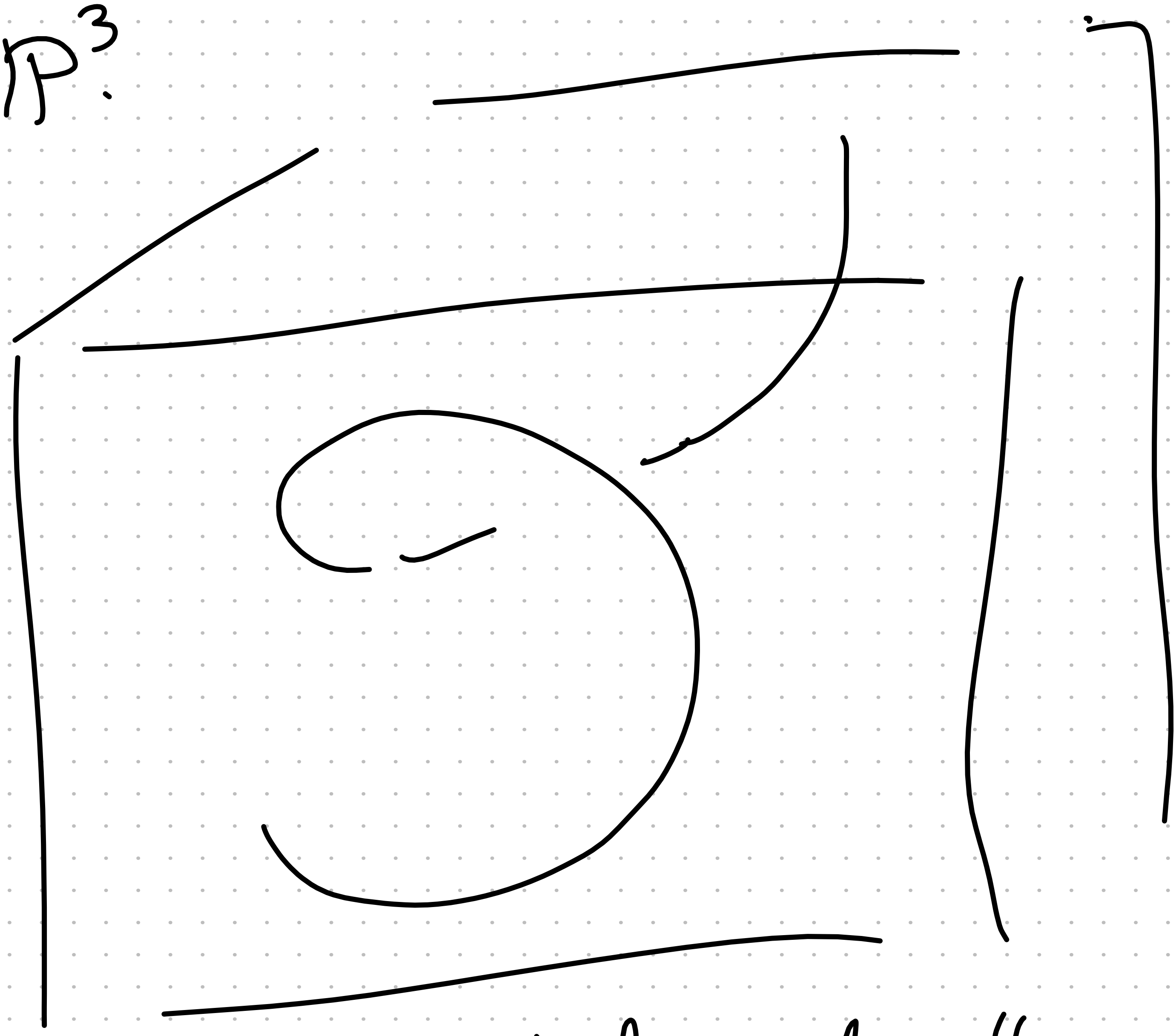
$$\mathbb{P}^1$$


"Conic"
$\mathbb{P}^2$
$\mathbb{P}^1$

$\underline{\underline{\mathbb{P}^1}} \longrightarrow \underline{\mathbb{P}^3}$ "Veronese embedding"

$\underline{[X^3 : X^2Y : XY^2 : Y^3]}$

is an iso onto its image, which is a closed sub. of $\mathbb{P}^3$.



"Twisted cubic"