1. Segre embedding
2. Quadratic forms
3. $\mathbb{P}^2$ vs $\mathbb{P}^1 \times \mathbb{P}^1$
4. Separated-ness

Segre embedding

$$[X_i], [Y_j] \mapsto \underbrace{\begin{bmatrix} X_0Y_0 & X_0Y_1 & \cdots & X_0Y_m \\ \vdots & & & \\ X_nY_0 & \cdots & \cdots & X_nY_m \end{bmatrix}}$$

$$\mathbb{P}^n \times \mathbb{P}^m \rightleftarrows Z = \text{Rank 1 matrices} / \text{scaling}$$

$$(\text{col}M, \text{row}M) \longleftarrow M = (m_{ij})$$

any column non-zero

This is the inverse

$$\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3 \qquad \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

Homog. poly

$$\text{Image} = V(\underbrace{XW - YZ}_{\text{quadratic form}})$$

Quadratic Forms :

$$X_0, \dots, X_n \qquad V = \underbrace{k\langle X_0, \dots, X_n \rangle}_{(n+1)\text{-dim v.space}}$$

$$q(X) = \sum_{i \le j} a_{ij} X_i X_j$$

(Char $k \neq 2$)

$$= \sum_{\substack{i,j \\ i \neq j}} \frac{a_{ij}}{2} X_i X_j + \sum a_{ii} X_i^2$$

$$= X^T A X \qquad A = \begin{pmatrix} a_{00} & & a_{ij}/2 \\ & \ddots & \\ a_{ij}/2 & & a_{nn} \end{pmatrix}$$

symmetric matrix.

$q(\underline{x}) = \underline{x}^T \underline{\underline{A}}\, \underline{x}$

symmetric.

Associated symmetric $\langle , \rangle$

$\langle x, y \rangle = x^T A y$

symmetric inner product

<u>Linear algebra</u> $\Rightarrow$ Any symmetric inner product can be <u>diagonalised</u>

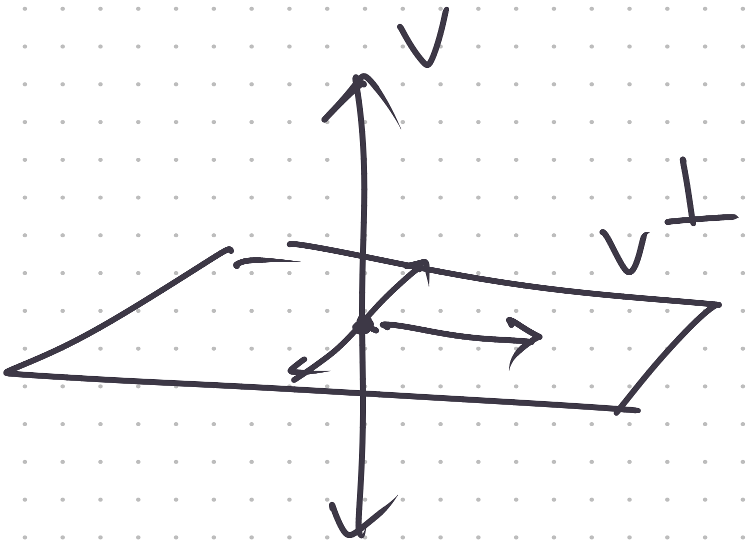
i.e. $\exists$ basis of the vector space such that in this basis the form is

$\langle x, y \rangle = x^T \underline{D}\, Y$

<u>Proof</u>: If $\langle , \rangle \equiv 0$ then done

otherwise $\exists\ \underline{v}$ with $\langle v, v \rangle \neq 0$

$$v + \underbrace{\text{orth complement of } v}_{\text{one less dim}} = V$$



Gram schmidt

$V_0, V_1, \ldots, V_n$ satisfying

$\langle V_i, V_j \rangle = 0$ for $i \neq j$

$V = \sum x_i V_i$

$w = \sum y_i V_i$

$\langle v, w \rangle = \sum x_i y_i \langle V_i, V_i \rangle$

↳ matrix $\begin{pmatrix} \langle v_0, v_0 \rangle & & \\ & \ddots & \\ & & \langle v_n, v_n \rangle \end{pmatrix}$

Any form up to change of basis is eqv to a diagonal one

$$q(x) = \sum \underline{\underline{\lambda_i}} \, x_i^2$$

over any field (char $\neq 2$)

<u>k alg closed</u>

$$q(x) = \sum (\underbrace{\sqrt{\lambda_i}\, x_i}_{\text{new variable}})^2$$

$$\boxed{x_0^2 + \cdots + x_\ell^2}$$

finite <u>list</u>

$0 \leq \ell \leq n$

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}$$ rk (QH)

Given a q, how do I tell (quickly) which one it is equivalent to?

$$q(x) = x^T A x$$

Change of basis $x \mapsto Bx$

B an invertible matrix

$$x^T A x \rightsquigarrow x^T \underbrace{B^T A B} x$$

$$A \rightsquigarrow \underline{B^T A B}$$

$\exists B$ s.t. $B^T A B$ is diagonal.

$$\operatorname{rk}(A) = \operatorname{rk}(B^T A B)$$

$$q(x) = X_0X_3 - X_1X_2$$

matrix

$$A = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} & & & 1/2 \\ & & -1/2 & \\ & -1/2 & & \\ 1/2 & & & \end{pmatrix} \end{array}$$

$(i,j)^{th}$ entry $= \frac{1}{2} \cdot$ coeff of X_iX_j

$$rk(A) = 4$$

$\Rightarrow q$ is non-deg. i.e.

$$\sim X_0^2 + X_1^2 + X_2^2 + \lambda X_3^2$$

$\lambda \to 0$

Quadratic forms / change of basis

$\cup$

Non-deg. forms are Zariski open set

Non deg q. forms / change of basis

$\|$

(n+1) vars

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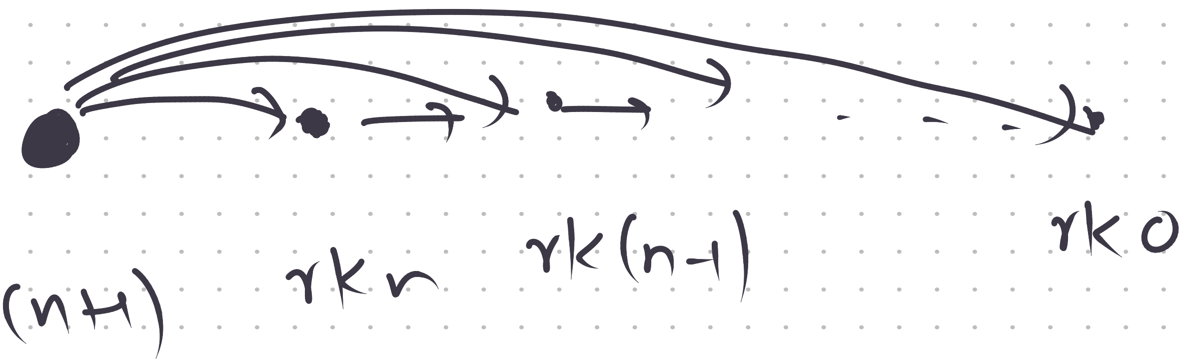
All q forms / change of basis



Cubic forms / change of basis

on $\mathbb{P}^1$ ie 2 vars.

$\longrightarrow$ finite

$\mathbb{P}^2 \longrightarrow$ 3 vars

Quotient is a 1-param space

is 1 dim.

$\mathbb{P}^3 \longrightarrow$ 4 vars

$\hookrightarrow$ 4 param space

$\mathbb{P}^1 \times \mathbb{P}^1$ vs $\mathbb{P}^2$ not iso $\quad (q+1)^2 \neq q^2+q+1$

$\downarrow$ admits but $\nexists$ reg map $\Leftarrow$

$\mathbb{P}^1 \qquad\qquad \mathbb{P}^2 \to \mathbb{P}^1$

(non-const)

Over $\mathbb{C}$

$$\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \simeq \mathbb{P}^2$$

Key: $f: X \to Y$ reg map

X & Y also have the usual top

$f: X \to Y$ continuous also in usual top

Reg iso $\Rightarrow$ homeomorphism (in the usual top)

$\mathbb{CP}^1 \times \mathbb{CP}^1 \not\cong \mathbb{CP}^2$ NOT <u>HOMEO</u>

<u>One proof</u> Homology is different

$H_2(\mathbb{CP}^2) \cong \boxed{\mathbb{Z}}$

$H_2(\mathbb{CP}^1 \times \mathbb{CP}^1) \cong \boxed{\mathbb{Z} \times \mathbb{Z}}$

<u>over k</u> (<u>alg</u> <u>closed</u>)

"Algebraic <u>homology</u> <u>theory</u>."

<u>Chow</u> groups

$A_1(X) = \{\text{curves} \subset X\} / \text{eqv}$

dim 1 <u>closed</u> sub

$C \sim C'$

$\underline{\text{Ex}}$ $\mathbb{P}^2 \supset C$

$C = V(F)$ F of deg $\underline{\underline{d}}$

$C' = V(G)$ G of deg $\underline{\underline{d}}$

$$V(\lambda \underline{\underline{F}} + (1-\lambda)\underline{\underline{G}}) \qquad \underline{\underline{\lambda \in k}}$$

$\lambda = 1$

$\lambda = 0$

on $\mathbb{P}^2$, this eqv. reduces to the degree

$$\underline{\underline{A_1(\mathbb{P}^2) \cong \mathbb{Z}}}$$

$\mathbb{P}^1 \times \mathbb{P}^1 \rightsquigarrow$ bi-degrees

$$A_1(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z} \times \mathbb{Z}$$

Gives a proof that

$$\mathbb{P}^2 \not\cong \mathbb{P}^1 \times \mathbb{P}^1$$

Separatedness $\Longleftrightarrow$ Hausdorff

$\hookrightarrow$ All quasi-proj var are

$X \to X \times X$ closed image

$\mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$

Image $= V$(bi-hom. system)

$$\begin{array}{ccc} X \subset \mathbb{P}^n & & \\ \varphi \downarrow & \searrow \psi & \\ X \times X \subset \mathbb{P}^n \times \mathbb{P}^n & & \end{array}$$

$\text{Im}(\varphi) = X \times X \cap (\text{Im}\, \psi)$

$(\text{Im}\, \psi) \subset \mathbb{P}^n \times \mathbb{P}^n$ closed.

$\Rightarrow \text{Im}(\varphi) \subset X \times X$ is.