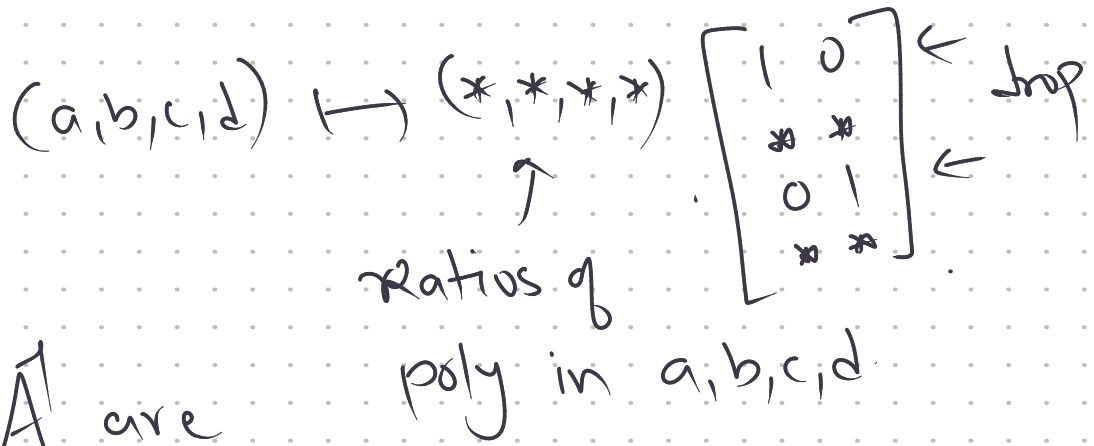
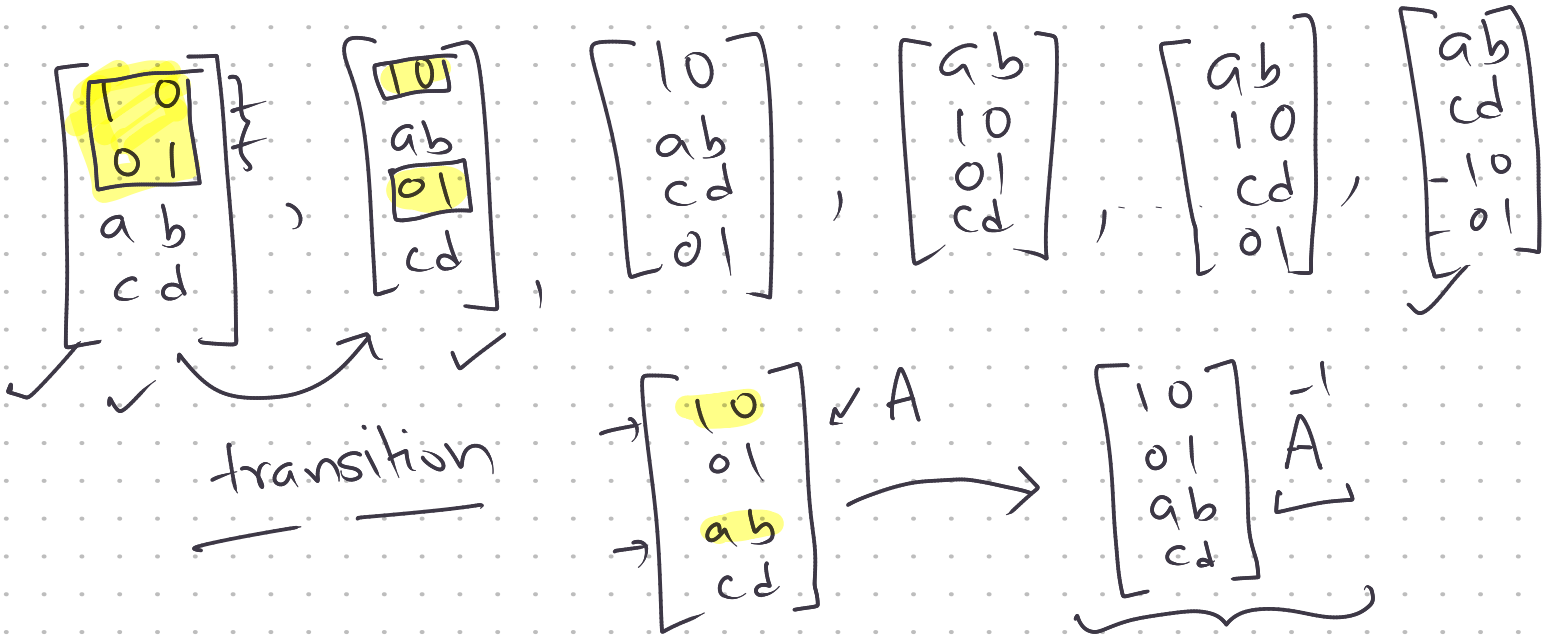


Grassmannian Gr(2,4)

6 charts $\cong \mathbb{A}^4$



Key:

Entries of A^{-1} are

poly in entries of A . ✓

$\frac{\text{poly}}{\det(A)}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ a & b \\ c & d \end{bmatrix} \xrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}} \begin{bmatrix} * & * \\ * & * \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \\ 0 & 1 \end{bmatrix} \leftarrow \text{drop}$$

$(a, b, c, d) \rightarrow$ entries of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$

Plücker map

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ a & b \\ c & d \end{bmatrix}$$

choose chart that drops 0^{th}

$$\mapsto [\underline{1} : b : d : -a : -c : ad - bc]$$

chart

Chart in the domain

$$(b, d, -a, -c, ad - bc)$$

polynomials

$$X \rightarrow Y$$

\downarrow 2 open

\downarrow 2 open

$$U_I \dashrightarrow V$$

$\Rightarrow p$ is regular

Closed embedding

$$p: X \rightarrow Y$$



① $\text{Im}(p) \subset Y$ is closed

② $p: X \rightarrow \text{Im}(p)$ isomorphism

Easy fact:

$$p: X \rightarrow Y$$

any map

p is a closed embedding iff

for some (equiv. for every) open

cover $\{U_i\}$ of Y , the maps

$p: p^{-1}(U_i) \rightarrow U_i$ are closed embeddings

Slogan :

"Being a closed embedding is local on the target."

$$p: X \rightarrow Y$$

↪ Can be checked after passing to an open cover of the target.

$$I \in \binom{4}{2}$$

$$\text{Gr}(2,4) \xrightarrow{p} \mathbb{P}^6$$

Choose the std open cover of \mathbb{P}^6

$$V_I = \{x_I \neq 0\} \quad I = \{1,2\}$$

$$p^{-1}(V_I) = \cup_I = \left\{ \begin{bmatrix} \boxed{\text{invertible}} \\ v \end{bmatrix} \right\}$$

$$\text{Gr}(2,4) \xrightarrow{p} \mathbb{P}^6$$

\cup

\cup

$$\{\det V_{12} \neq 0\} \longrightarrow \{x_{12} \neq 0\}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & b \\ c & d \end{pmatrix} \xrightarrow{p} (b, d, a, -c, ad-bc)$$

$(a, x_2, -x_5)$

$\mathbb{A}^4 \xrightarrow{\quad} \mathbb{A}^5$

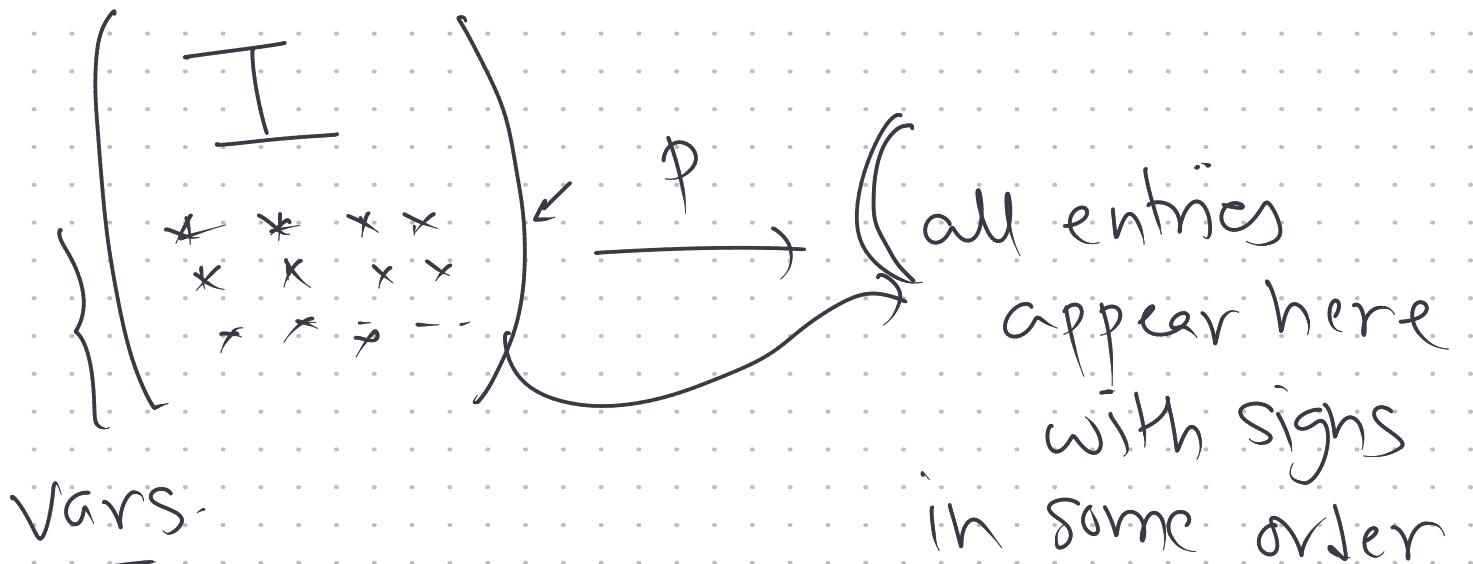
Q: Closed embedding?

A: YES

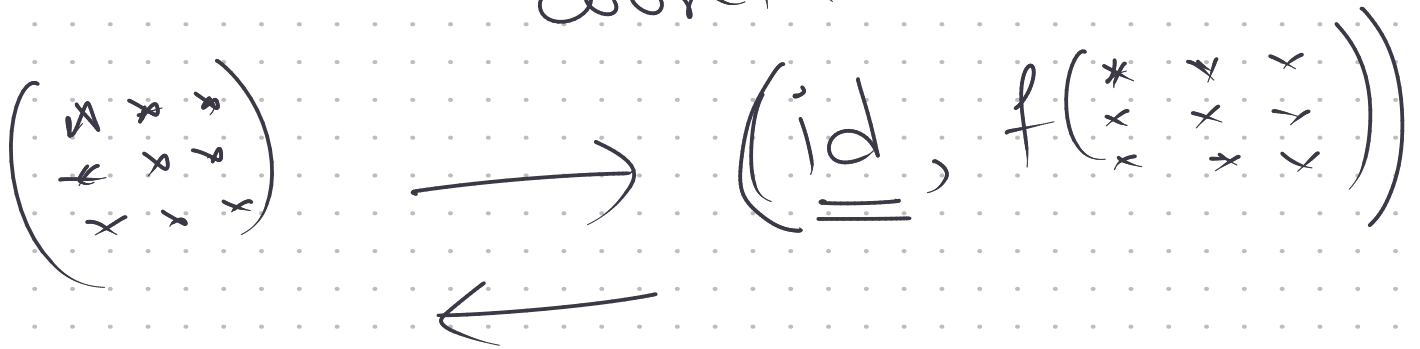
$$\text{Im}(p) = V(x_5 - x_2(-x_3) + x_1(-x_4))$$

$(x_1, -x_5) \mapsto (-x_3, x_1, -x_4, x_2)$

Inverse \downarrow
 \mathbb{A}^4



\leftarrow
 proj on the first few coordinates



$$\text{Gr}(m, n) \cong \text{Gr}(n-m, n)$$

Choose $V \cong \mathbb{k}^n$ $V^\vee = \text{Hom}(V, \mathbb{k})$

There is a canonical iso.

$$\text{Gr}(m, V) \cong \text{Gr}(n-m, V^\vee)$$

$$\boxed{W \subset V} \xrightarrow{\quad} \begin{array}{c} n \\ V \rightarrow V/W \end{array} \quad \swarrow \text{dual}$$
$$\hookrightarrow \boxed{(V/W)^\vee \hookrightarrow V^\vee}$$

$W \subset V$ of dim m

Then W is cut out by $(n-m)$ linear functions on V

$W \mapsto$ Span of linear functions vanishing on $W \subset V$

$$\text{Gr}(m, V) \xrightarrow{\sim} \text{Gr}(n-m, V^{\vee})$$

$$\text{Gr}(1, V) \xrightarrow{\sim} \text{Gr}(n-1, V^{\vee})$$

$\Rightarrow \parallel$
IP V

\uparrow
iso. to a
proj. space

$$\boxed{\text{Gr}(2, 4)}$$

Varieties over fields other than alg closed ones

Philosophy :-



Equations live over Q

But

solutions always in Q

Q, Affine var / Q.

By def it is $X \subset \mathbb{A}^n_{\mathbb{Q}}$ \swarrow Sols

of the form $V(f_1, \dots, f_m) \cap g_j \neq 0$

$f_i \in \mathbb{Q}[x_1, \dots, x_n]$ $g_j \in \mathbb{Q}[\dots]$

X, Y defined over \mathbb{Q} .

$f: X \rightarrow Y$ defined over \mathbb{Q}

$= (f_1, \dots, f_m)$

$\hookrightarrow \frac{\text{poly}}{\text{poly}} \in \mathbb{Q}[\dots]$

Variety / \mathbb{Q} is one with
charts & transitions def.
over \mathbb{Q}

Ex ✓ $\mathbb{A}_{\mathbb{Q}}^n$ is defined over \mathbb{Q} ,

✓ $\mathbb{P}_{\mathbb{Q}}^n$ is def over \mathbb{Q} ,

✓ $\text{Gr}(n, m)$ def over \mathbb{Q} ,

Segre:

$$\mathbb{P}^1 \times \mathbb{P}^1 \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \uparrow \end{array} V(XY - ZW) \subset \mathbb{P}^2$$

defined over \mathbb{Q} .

$$V(x^2 + y^2 + z^2 + w^2) \stackrel{\cong}{=} V(XY - ZW) \stackrel{\cong}{=} \mathbb{Q}(i)$$

NOT defined over \mathbb{Q} .
 \exists over $\mathbb{Q}(i)$

$$[x:y] \mapsto [ix:iy]$$
$$\mapsto [x:y] \checkmark$$

$$\mathbb{Q} \subset \mathbb{Q}(i) \subset \mathbb{Q}(i)$$

How do you prove def over \mathbb{Q}

$$V(\underline{X^2 + Y^2 + Z^2 + W^2}) \neq V(\underline{XY - ZW}) \text{ over } \underline{\mathbb{Q}}$$

X defined over \mathbb{Q}
= make sense to talk about
pts of X defined over \mathbb{Q}

$$\boxed{X(\mathbb{Q})}$$

$$\| \begin{matrix} x \in X \\ (x_1, \dots, x_n) \end{matrix} \quad x_i \in \mathbb{Q}$$

in some chart \Leftrightarrow (any chart)

$$\textcircled{f}: X \xrightarrow{\sim} Y \text{ over } \underline{\mathbb{Q}}$$
$$f: X(\mathbb{Q}) \xrightarrow{\sim} Y(\mathbb{Q})$$

All non deg quadrics are

ISO

↑
only over alg. closed field

Arithmetic geometry

\mathbb{Z}

↳ Alg. geometry over
a base field k which is

() not alg. closed

() base ring \mathbb{R} , which may
not be an alg. closed field.

$$V(X^2 + Y^2 + Z^2 + W^2) = X \subset \mathbb{P}^3$$

$$V(XY - ZW) = Y \subset \mathbb{P}^3$$

$$X(\mathbb{Q}) = \emptyset$$

$$\boxed{X(\mathbb{Q}) = \emptyset}$$

$$X(\mathbb{R}) = \emptyset$$

$$\boxed{X \neq \emptyset}$$

$Y(\mathbb{Q})$ tons.

$\Rightarrow X \not\cong Y$
over \mathbb{Q} .

$$X \subset \mathbb{P}_{\mathbb{Q}}^3$$

$$\overline{\mathbb{Q}} = \mathbb{Q}$$