Grassmannian $Gr(2,4)$

6 charts $\cong \mathbb{A}^4$

$$\begin{bmatrix} \boxed{\begin{matrix}1 & 0\\ 0 & 1\end{matrix}} \\ \begin{matrix}a & b\\ c & d\end{matrix} \end{bmatrix}, \quad \begin{bmatrix} \boxed{1 \; 0} \\ a \; b \\ \boxed{0 \; 1} \\ c \; d \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ a & b \\ c & d \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} a & b \\ 1 & 0 \\ 0 & 1 \\ c & d \end{bmatrix}, \quad \begin{bmatrix} a & b \\ 1 & 0 \\ c & d \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} a & b \\ c & d \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

✓ ✓ ✓ ✓

transition

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ a & b \\ c & d \end{bmatrix} A \longrightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ a & b \\ c & d \end{bmatrix} A^{-1}}$$

$$(a,b,c,d) \longmapsto (*,*,*,*) \qquad \begin{bmatrix} 1 & 0 \\ * & * \\ 0 & 1 \\ * & * \end{bmatrix}$$

drop

ratios of poly in a,b,c,d

Key:

Entries of A^{-1} are $\dfrac{\text{poly}}{\det(A)}$ in entries of A. ✓

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ a & b \\ c & d \end{bmatrix} \xrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}} \begin{bmatrix} * & * \\ * & * \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \\ \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{drop} \\ \leftarrow \end{matrix}$$

$$(a, b, c, d) \to \text{entries of } \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$$

Plücker map

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ a & b \\ c & d \end{bmatrix} \mapsto [1 : b : d : -a : -c : ad-bc]$$

choose chart that drops 0th

$$\downarrow \text{chart}$$

$$(b, d, -a, -c, ad-bc)$$

$$\uparrow$$

polynomials

Chart in the domain

$$\begin{tikzcd} X \arrow[r] & Y \\ U_I \arrow[u, "\supseteq \text{open}"] \arrow[r, dashed] & V \arrow[u, "\supseteq \text{open}"'] \end{tikzcd}$$

$\Rightarrow$ p is regular

Closed embedding

$p : X \to Y$ $\Longleftrightarrow$ ① $\mathrm{Im}(p) \subset Y$ is closed

② $p : X \to \mathrm{Im}(p)$ isomorphism

Easy fact:

$p : X \to Y$ any map

p is a closed embedding iff for some (eqv. for every) open cover $\{U_i\}$ of Y, the maps

$$p : p^{-1}(U_i) \to U_i$$ are closed embeddings.

Slogan:

"Being a closed embedding is local on the target."

↳ Can be checked after passing to an open cover of the target.

$p : X \to Y$

$I \in \binom{4}{2}$

$Gr(2,4) \xrightarrow{p} \mathbb{P}^6$

Choose the std open cover of $\mathbb{P}^6$

$V_I = \{ x_I \neq 0 \}$ $\quad I = \{1,2\}$

$p^{-1}(V_I) = U_I = \left\{ \begin{bmatrix} \boxed{} \\ V \end{bmatrix} \right.$ ← invertible $\left. \right\}$

$$\begin{array}{ccc} Gr(2,4) & \xrightarrow{p} & \mathbb{P}^6 \\ \cup & & \cup \\ \{\det V_{12} \neq 0\} & \longrightarrow & \{X_{12} \neq 0\} \end{array}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & b \\ c & d \end{pmatrix} \overset{p}{\longmapsto} (\boxed{b, d, -a, -c}, ad-bc)$$

$(x_1, x_2, \dots, x_5)$

$$\mathbb{A}^4 \rightleftarrows \mathbb{A}^5$$

Q: Closed embedding?

A. YES

$$Im(p) = V\big(\underline{x_5} - \underbrace{x_2 \cdot (-x_3) + x_1 \cdot (-x_4)}\big)$$

Inverse $\downarrow$ $\quad (x_1, \dots, x_5) \mapsto (-x_3, x_1, -x_4, x_2)$

$\mathbb{A}^4$

$$\text{vars}\left\{\begin{pmatrix} I \\ * \; * \; * \; * \\ * \; * \; * \; * \\ * \; * \; * \; \cdots \end{pmatrix}\right. \xrightarrow{\;p\;} \Big(\text{all entries appear here with signs in some order, poly of the entries}\Big)$$

$$\xleftarrow{\quad\quad}$$

proj on the first few coordinates

$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \longrightarrow \left(\underline{\underline{id}},\ f\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}\right)$$

$$\longleftarrow$$

$$Gr(m,n) \cong Gr(n-m,n)$$

Choose $V \cong k^n$ $\quad V^\vee = Hom(V,k)$

There is a canonical iso:

$$Gr(m,V) \cong Gr(n-m, V^\vee)$$

$\boxed{\overset{}{W} \subset \overset{}{V}}$ $\quad \overset{n}{V} \twoheadrightarrow \overset{n-m}{V/W}$ $\quad$ ✓ dual

$\hookrightarrow \boxed{(V/W)^\vee \hookrightarrow V^\vee}$

$\overset{m}{W} \subset \overset{n}{V}$ of dim m

Then W is cut out by $(n-m)$ linear functions on V

$W \longmapsto$ Span of linear functions vanishing on $W \subset V^\vee$

$$Gr(m, V) \xrightarrow{\sim} Gr(n-m, V^\vee)$$

$$Gr(1, V) \xrightarrow{\sim} Gr(n-1, V^\vee)$$

$\Rightarrow$ $Gr(1,V) = \mathbb{P}V$

$Gr(n-1, V^\vee)$ iso to a proj. space $\Leftarrow$

$$\boxed{Gr(2,4)}$$

Varieties over fields other than alg closed ones

$\mathbb{Q}$

Philosophy :-

Equations live over $\mathbb{Q}$

But

solutions always in $\overline{\mathbb{Q}}$

Q. Affine var / $\mathbb{Q}$.

Sol^{n}s

By def it is $X \subset \mathbb{A}^n_{\overline{\mathbb{Q}}}$

of the form $V(f_1, \dots, f_m) \cap g_j \neq 0$

$f_i \in \mathbb{Q}[x_1, \dots, x_n]$ $\quad g_j \in \mathbb{Q}[\dots]$

X, Y defined over $\mathbb{Q}$.

$f : X \to Y$ defined over $\mathbb{Q}$

$= (f_1, \ldots, f_m)$

$\hookrightarrow \dfrac{\text{poly}}{\text{poly}} \in \mathbb{Q}[\ldots]$

Variety $/\mathbb{Q}$ is one with charts & transitions def <u>over</u> $\mathbb{Q}$

<u>Ex</u> ✓ $\mathbb{A}^n_{\overline{\mathbb{Q}}}$ is defined over $\underline{\mathbb{Q}}$,

✓ $\mathbb{P}^n_{\overline{\mathbb{Q}}}$ is def over $\underline{\mathbb{Q}}$,

✓ $Gr(n, m)$ def over $\mathbb{Q}$.

Segre :

$$\mathbb{P}^1 \times \mathbb{P}^1 \rightleftarrows V(XY - ZW) \subset \mathbb{P}^2$$

defined over $\mathbb{Q}$

$V(X^2 + Y^2 + Z^2 + W^2)$

$\cong V(XY - ZW)$

$\boxed{\text{NOT}}$ defined over $\mathbb{Q}$

IS over $\overline{\mathbb{Q}}$

$\mathbb{Q}(i)$ $=$

$\overline{\mathbb{Q}} \cup F \cup \mathbb{Q}$

$[X:Y] \mapsto [iX : iY]$

$\mapsfrom [X:Y]$ ✓

How do you prove

$V(X^2+Y^2+Z^2+W^2)$ ← def over $\mathbb{Q}$

$\not\cong V(XY-ZW)$ over $\mathbb{Q}$

X defined over $\mathbb{Q}$

= make sense to talk about

pts of X defined over $\mathbb{Q}$

$\parallel$

$\boxed{X(\mathbb{Q})}$

$x \in X$

$\parallel$

$(x_1, \ldots, x_n)$ $x_i \in \mathbb{Q}$

in some chart $\iff$ (any chart)

$f : X \xrightarrow{\sim} Y$ over $\mathbb{Q}$

$f : X(\mathbb{Q}) \xrightarrow{\sim} Y(\mathbb{Q})$

All nondeg quadrics are

ISO

only over alg. closed field

Arithmetic geometry $\mathbb{Z}$

↳ Alg geometry over a base field k which is not alg. closed

↳ base ring R, which may not be an alg. closed field.

$$V(X^2+Y^2+Z^2+W^2) = X \subset \mathbb{P}^3$$

$$V(XY-ZW) = Y \subset \mathbb{P}^3$$

$$X(\mathbb{Q}) = \emptyset \qquad \boxed{X(\mathbb{Q}) = \emptyset}$$

$$X(\mathbb{R}) = \emptyset \qquad \boxed{X \neq \emptyset}$$

$Y(\mathbb{Q})$ tons.

$$\Rightarrow X \not\cong Y \qquad \bar{\mathbb{Q}}$$

over $\mathbb{Q}$.

$$X \subset \mathbb{P}^3_{\bar{\mathbb{Q}}}$$