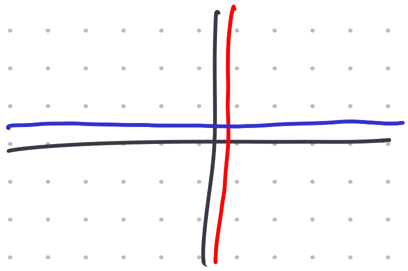
Properties of alg varieties

1. Irreducibility
2. Dimension
3. Tangent spaces and smoothness
4. Completeness

Irreducibility –

Example – Reducible → $V(xy) \subset \mathbb{A}^2$

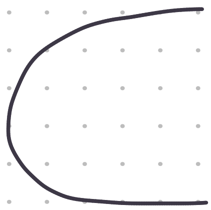
factors



Irreducible

$V(y^2 - x)$

irreducible



Def: X is reducible if we can write

$X = Y \cup Z \qquad Y, Z \subsetneq X$

closed

(x any top space)

X irreducible $\Leftrightarrow$ Not reducible.

(*) $X \subset \mathbb{A}^n$ is closed (i.e. X affine)

Then TFAE

1. X is irreducible
2. $k[X]$ is a domain — "integral domain"
3. $I(X)$ is prime.

Conseq.: $X = V(f)$ $f \in k[x_1, \dots, x_n]$

Then X is irreducible if f is irreducible

To prove (*), useful to know -

Prop: TFAE

1. X is irreducible
2. Any open $U \subset X$ (non-empty) is dense in X
3. Any two opens (non-empty) have a non-empty intersection

Pf: Let's see ① $\Leftrightarrow$ ③

①$\Rightarrow$③ X irreducible $\leftarrow$

Take $U, V \subset X$ $U \neq \emptyset$, $V \neq \emptyset$

Want $U \cap V \neq \emptyset$.

Compute $(U \cap V)^c$

$$= \underbrace{U^c}_{\substack{\text{closed} \\ \subsetneq X}} \cup \underbrace{V^c}_{\substack{\text{closed} \\ \subsetneq X}} \overset{\text{irreducibility}}{\subsetneq} X$$

$\Rightarrow U \cap V \neq \emptyset$

③$\Rightarrow$① Same argument.

□

(A) Gives a characterisation of irreducible affine.

Two properties

① $U \subset X$ dense $\Rightarrow$
U is irred iff X is irred.

Conseq: The closure of an irred space is irreducible

② $f: X \to Y$ continuous surjection
X irred $\Rightarrow$ Y irreducible.

$\Rightarrow$ $\mathbb{P}^n$ is irreducible.

$$\begin{array}{ccc} \mathbb{A}^{n+1} \setminus \{0\} & \xrightarrow{\text{surj}} & \boxed{\mathbb{P}^n} \text{ irred} \\ \text{irred} \quad \big\downarrow \text{open} & & \\ \mathbb{A}^{n+1} & \leftarrow \text{irreducible} \checkmark & \end{array}$$

(Open subs of irred are irred.)

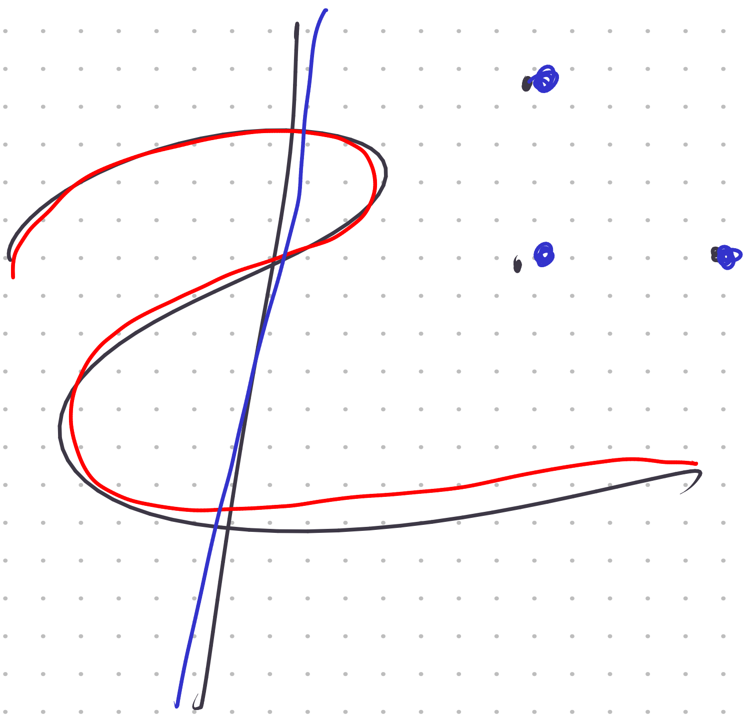
$$\begin{tikzcd} \{\text{Rank } m\} \arrow[r] \arrow[d, hook, "\text{open}"] & Gr(n,m) \\ \mathbb{A}^{n\times m} & \end{tikzcd}$$

$\leftarrow$ irred

$\uparrow$ irred.

$\mathbb{A}^{n\times m} \leftarrow$ irred

Reducible:



$X = Y \cup Z$

$X = Y \cup Z$

reducible?

No. Stop

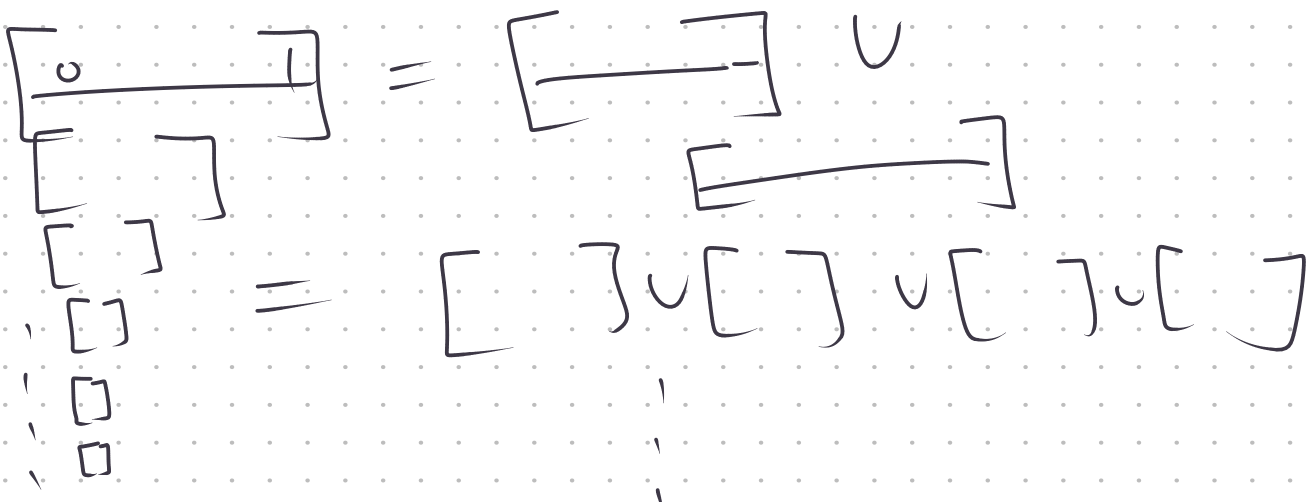
$= Y \cup L \cup 3$ pts

✓ ✓

$X = Y \cup L \cup pt_1 \cup pt_2 \cup pt_3$

Why should this terminate?

Ex. Euclidean top



This does not happen with Zariski top

The Zariski top on quasi-proj var is Noetherian.

Def: X Noetherian if every seq of closed subsets $Z_1 \supset Z_2 \supset \cdots$ stabilises.

$\underline{X}$ affine $\Rightarrow$ X Noetherian.

Why? $X \subset \mathbb{A}^n$ $I = I(X)$

$X = V(I)$

$X \supset Z_1 \supset Z_2 \supset \cdots\cdots$

$\rightsquigarrow$

$I_1 \subset I_2 \subset I_3 \subset \cdots\cdots$

$\cap$

$k[x_1, \ldots, x_n]$

$\rightsquigarrow$ stabilises (Noetherian ness of $k[x_1, \ldots, x_n]$

$\Rightarrow$ Z_i stabilises.

Easy - Finite union of Noeth spaces is Noetherian.

Any variety which is a union of finitely many affines is Noetherian

Any quasi-affine is a finite union of affines.

$\Rightarrow$ Any variety that has an atlas consisting of finitely many quasi-affine charts is Noetherian.

"finite type"

All varieties we've seen

Thm All varieties of finite type are Noetherian.

$X = X_1 \cup X_2$ if reducible

keep going.

This terminates, & get

$\cancel{X} = X_1 \cup X_2 \cup \cdots \cup X_n$

$X_i \subsetneq X$ closed irreducible

Assume $X_i \not\subseteq X_j$ for any i, j

Then this decomp is <u>unique</u>

up to re-ordering.

X_i "irreducible components"

X irreducible

$\Rightarrow$ Any two non-empty opens intersect.

$f : X \dashrightarrow Y$ a reg. fun on an open subset of X

Def: A rational map

$X \dashrightarrow Y$ is a reg map defined on some open subset of X
non-empty

More precisely, a rational map $X \dashrightarrow Y$ a pair (U, f) where $U \subset X$ non-empty open & $f : U \to Y$ reg.

(U, f) & (V, g) are considered the same if on $U \cap V$ ← non-empty!

you have $f = g$.

Example

$$X = V(xy - z^2) \subset \mathbb{A}^2$$

$f = \dfrac{x}{z}$ defined on

U = compl. of $z=0$

$X \dashrightarrow \mathbb{A}^1$ (U, f) ←

$X \dashrightarrow \mathbb{A}^1$ (V, g) ←

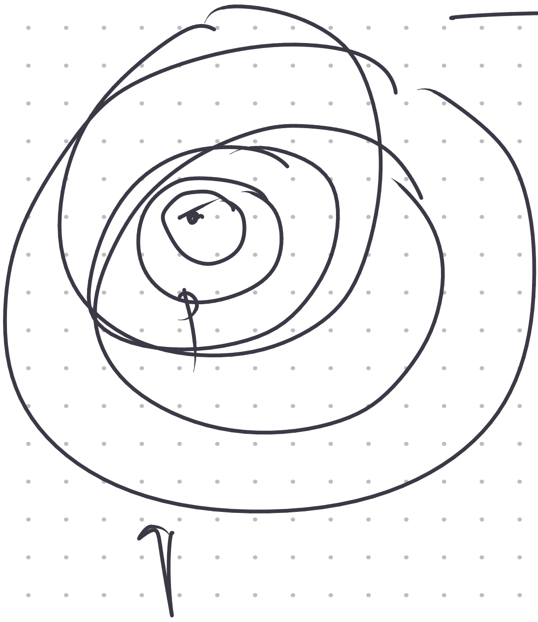
$g = \dfrac{z}{y}$ on

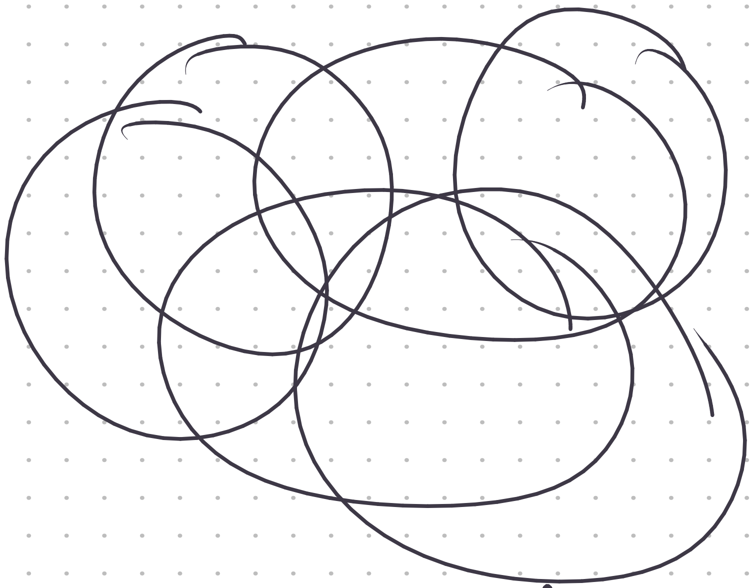
V = compl. of $y=0$.

But $\dfrac{x}{z} = \dfrac{z}{y}$ when they are both defined

Rat. fun = eqv. class of (U, f)

Germ at p (U, f) $p \in U$





Germ

concentrated at p.

rat fun

"diffused" all over X.

Rat map $f : X \dashrightarrow Y$

e.g.

$$\begin{tikzcd} \mathbb{P}^n \arrow[r, dashed] & \mathbb{A}^1 \\ \mathbb{A}^n \arrow[u, phantom, "\cup"] \arrow[ur, dashed, "f/g"'] & \end{tikzcd}$$

Rational maps $\mathbb{P}^n \dashrightarrow Y$

$\Updownarrow$

Rat maps from $\mathbb{A}^n \dashrightarrow Y$

$$\begin{tikzcd} \mathbb{P}^n \arrow[r, dashed] & Y \\ \mathbb{A}^n \arrow[u, phantom, "\cup"] \arrow[ur, dashed] & \end{tikzcd}$$

$$\mathbb{P}^n \ \underline{=}\ \underbrace{\mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{n}$$

Terminology $\mathbb{P}^n$, $\mathbb{A}^n$, $(\mathbb{P}^1)^n$

"birational" to each other.

Birational $\underline{X}$, $\underline{Y}$

if $f: X \dashrightarrow Y$

$g: Y \dashrightarrow X$

s.t. $f \circ g$ & $g \circ f \underset{\text{eqv.}}{\sim} id_X$ & id_Y.

Easy:- $X \subset \mathbb{P}^n$

$X = V(\text{linear or irred quadr})$

then X is birat to $\mathbb{P}^{n-1}$.

Q: $X = V(\text{cubic}) \subset \mathbb{P}^n$

$X \subset \mathbb{P}^2 \longrightarrow$ NO

$X \subset \mathbb{P}^3 \longrightarrow$ YES

$X \subset \mathbb{P}^4 \longrightarrow$ 1974 NO

$X \subset \mathbb{P}^5 \longrightarrow$??? (NO??)