# Irreducibility & rational maps

(1) a) $U \subset X$ dense

$U$ irred $\Leftrightarrow$ $X$ irred.

b) $f : X \to Y$ surj, $X$ irred $\Rightarrow$ $Y$ irred.

(2) $X$ affine $X \subset \mathbb{A}^n$ closed

$X$ irred $\Leftrightarrow$ $k[X]$ is a domain

$\Leftrightarrow$ $I(X)$ is prime,

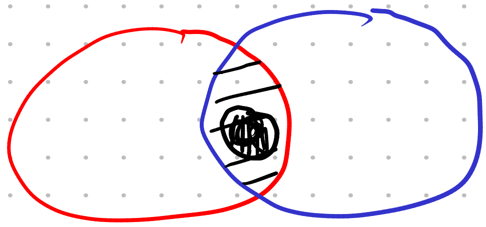
# Rational maps

$X$ irred, $Y$ separated.

A rational map is an eqv. class of $(U, f)$
$U \subset X$ open non-empty ($\Rightarrow$ dense)
& $f : U \to Y$ regular.

Relation: $(U, f) \sim (V, g)$ if
$f, g : U \cap V \to Y$ are equal.

Restatement: $(U, f) \sim (V, g)$ if there is
$\exists$ an non-empty open $W \subset U \cap V$ such that
$f, g : W \to Y$ are equal



Why are these equivalent?

HW problem:

$$\begin{tikzcd} \text{Domain} \arrow[r, "f,g"] & Y \\ \text{Dense} \arrow[u, phantom, "\cup"] \arrow[ur, "f=g"'] & \end{tikzcd}$$

separated

(*)

$$\begin{tikzcd} Y \times Y \\ Y \arrow[u, "\Delta"'] \end{tikzcd}$$

then $f=g$ on Domain.

Given this, $f,g : W \to Y$ equal then equal on $U \cap V$.

$W \subset U \cap V$

↳ irreducible

↳ Dense $\Leftarrow$

$(U,f) \sim (V,g) \sim (W,h)$

$(U,f) \sim (W,h)$ know

$f,h : U \cap V \cap W \to Y$ (*)

($\Rightarrow$ $f,h$ are equal on $U \cap W$)

Rational map ex.

Rat. fun := Rat map to $\mathbb{A}^1$. ✓ separated.

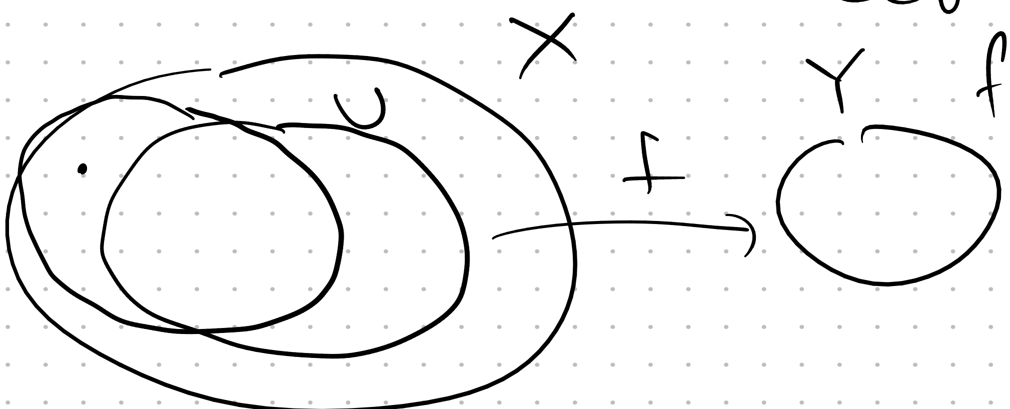
$X = \mathbb{P}^1 \quad \{[X:Y]\}$

$f = \frac{X}{Y}$ (U omitted)

$(U, \frac{X}{Y})$ $U \subset \mathbb{P}^1$ is $[X:Y]$, $Y \neq 0$.

$\frac{X^2+Y^2}{XY}$. $\left(\frac{\text{Homog}}{\text{Homog}}\right)$ Rat. fun on $\mathbb{P}^n$. same deg.

Domain of def.




define a rat map $f : X \dashrightarrow Y$

$(U, f)$

f is "undefined" at every $x \notin U$.

f is defined at $x \in X$ if $\exists$ $(V, g) \sim (U, f)$ s.t. $x \in V$

$X = V(xy - zw) \subset \mathbb{A}^4$

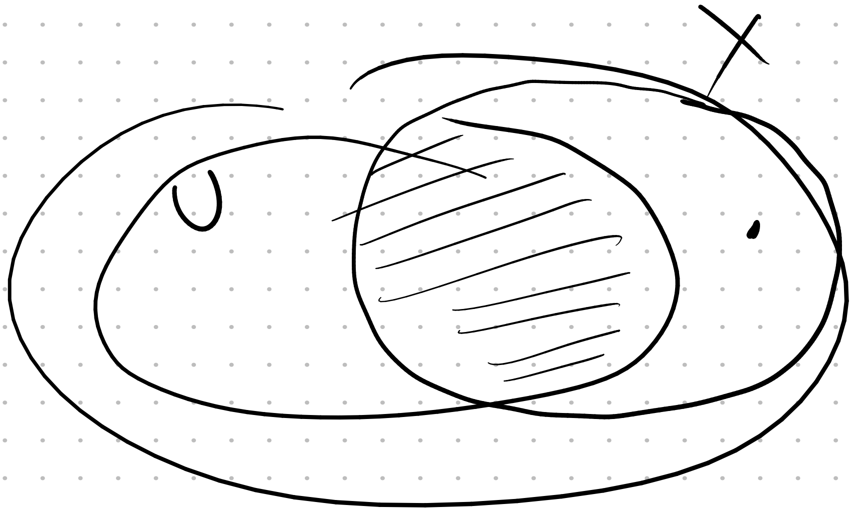
$f = \dfrac{x}{w}$ $\quad (U, f) \quad U = \{w \neq 0\} \cap X$

Is f defined at $(0,1,0,0)$

Can f be defined at $(0,1,0,0)$ ?

Can you find $(V, g) \sim (U, f)$

s.t. $(0,1,0,0) \in V$. $\qquad g = \dfrac{z}{y}$





$\underbrace{f = g}$ on the int. of domains.

$\left(w \neq 0, \dfrac{x}{w}\right) \sim \left(y \neq 0, \dfrac{z}{y}\right)$

$(0,1,0,0) \in$ Domain of def of rat map rep. by $f$.

(5) Field of rat fun.

$k(X)$ = Set of rat functions on $X$.

Ring $\quad (U, f)\ , (V, g) \qquad k$ algebra.

$+ \quad (U \cap V, f+g) \leftarrow (X, c)$

$\cdot \quad (U \cap V, fg) \qquad k \to k(X)$

eg. On $\mathbb{P}^1$ $\quad \frac{X}{Y} = f \qquad Y \neq 0$

$\frac{Y-X}{Y} = g \qquad Y \neq 0$

$f+g = 1$ defined everywhere.

$(Y \neq 0, 1) \sim (\mathbb{P}^1, 1)$

Field :- $(U, f)$ non-zero

$(X, 0)$ : zero element.

$(X, 0) \nsim (U, f)$ i.e. on $U$

the 0 fun & fun f are not equal.

$f \not\equiv 0$ on $U$.

Consider $V = U \setminus \{u \in U \mid f(u) = 0\}$.

↳ open in $U$

$V = U \setminus f^{-1}(0)$

$(V, 1/f)$ represents the mult. inv. of $(U, f)$.

$X \subset \mathbb{A}^n$ closed.

$$\operatorname{frac} k[X] \cong k(X)$$

$$\frac{f}{g}, \quad f, g \in k[X] \quad g \neq 0.$$

$$g : X \to \mathbb{A}^1$$

$$\frac{f}{g} \mapsto (U, f/g)$$

$$U = X - g^{-1}(0)$$

Clearly a ring homomorphism.

Domain a field $\Rightarrow$ injective.

Surjectivity: $(U, f) \in k(X)$

Take $u \in U$. $f$ is reg at $u$.

$\exists$ open $W \subset U$ containing $u$ such that

$$f = \frac{p}{q} \quad p, q \in k[\mathbb{A}^n]$$

$$p \mapsto \bar{p} \quad k[\mathbb{A}^n] \to k[X]$$

$\frac{\bar{p}}{\bar{q}} \in \operatorname{frac} k[X]$

is mapped to the rat. fun $(W, p/q)$.

# Irreducibility

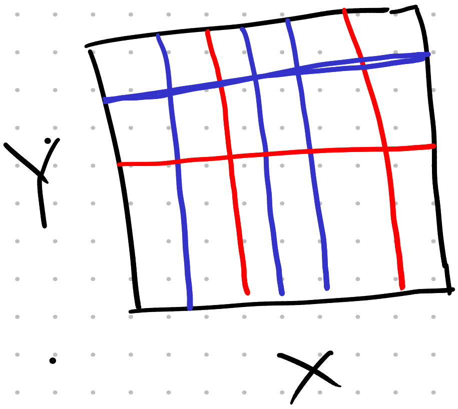
**Prop:** $X, Y$ irreducible then
$X \times Y$ is irreducible.

**Pf:**

$$X \times Y = Z_1 \cup Z_2$$

Two closed sets.

want $Z_1 = X \times Y$ or
$Z_2 = X \times Y$




$$(\underline{\{x\} \times} Y) = Z_1 \cap (\_) \cup Z_2 \cap (\_)$$

↑
irred

So the union is trivial, one of the factors is whole space.

Blue = Contained in $Z_2$

Red = Not contained in $Z_2$
$\Rightarrow$ contained in $Z_1$

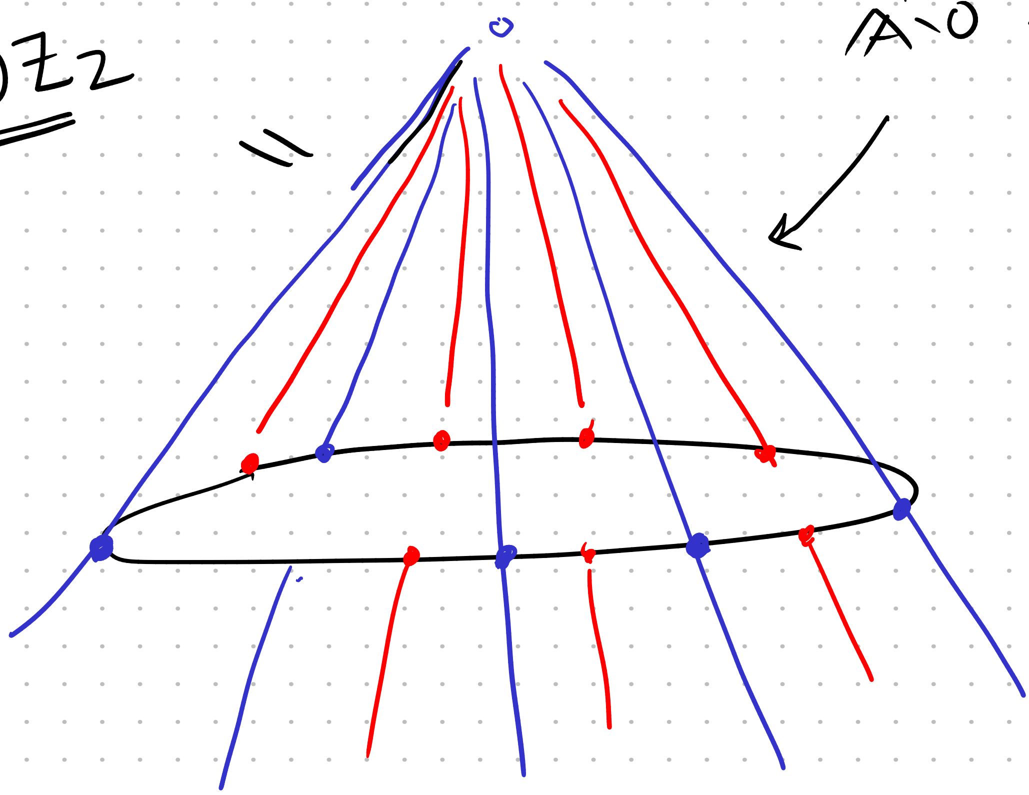
Doing this criss-cross $\Rightarrow$ all of $X \times Y$ must be in $Z_1$ or $Z_2$.
ie. $Z_1 = X \times Y$ or $Z_2 = X \times Y$.

**Prop:** $X \subset \mathbb{P}^n$ irred.

then $C_X \subset \mathbb{A}^{n+1} \setminus \{0\}$ is irred.

$\underline{Z_1 \cup Z_2}$ =

$\mathbb{A}^1 \setminus 0 \leftarrow$ irred.



$X$ =

either all lines are blue or all lines are red.

## Tools to show irreducibility

1. Images of irred.
2. Dense subsets.
3. Closures of irred.
4. Products
5. Cones.
6. Affines $\Leftrightarrow$ Domain.

surj

$f : \underline{X} \to Y$

$\Rightarrow$ -

$f : X \to Y$

$\mathrm{Im}(f)$

In general, showing that something is irred. can be hard.