Dimension

① $X \subset \mathbb{A}^n$ $\quad X = V(f)$ $\quad f \neq 0$

a) Then X is equidim of codim 1.

Pf: Pick $x \in X \subsetneq \mathbb{A}^n \leftarrow$ dim n

$\dim_x X \leq n-1$. $\leftarrow$ (Krull dim)

Because $X = V(f)$, principal ideal thm

$\Rightarrow$ $\dim_x X \geq n-1$ $\leftarrow$ (slicing dim)

Together get $\dim_x X = n-1$.

Holds for any irred X.

① b) If X is equidim of codim 1

then $X = V(f)$.

X equidim $\Rightarrow$ every irred comp of X is dim $n-1$

$X = X_1 \cup \cdots \cup X_r$ $\quad X_i$ irred dim $n-1$.

Suppose $r=1$ (i.e. X irred).

Consider $X = I(X)$

principal ?

$f \in I(X)$

$f = f_1 \cdots f_m$ $\quad f_i$ irred. poly.

$f \in I(X) \leftarrow$ prime

$f_i \in I(X)$ for some i.

$X \subset V(f_i) \leftarrow$ irreducible

f_i irred & $k[x_1, \ldots, x_n]$ a UFD

$\Rightarrow (f_i)$ is prime

$X \subset V(f_i) \leftarrow$ irred dim $(n-1)$

dim $(n-1)$ $\Rightarrow X = V(f_i)$

$$\begin{array}{cccc} X = & X_1 \cup & \cdots & \cup X_r \\ & \| & & \| \\ & V(f_1) & & V(f_r) \end{array}$$

$$X = V(\underbrace{f_1 \cdots f_r}_{\text{product}})$$

If X is an irred affine & $k[X]$ is a <u>UFD</u> then any (equi)-codim 1 closed suvariety $Y \subset X$ is of the form $V(f)$ for some $f \in \underline{k[X]}$.

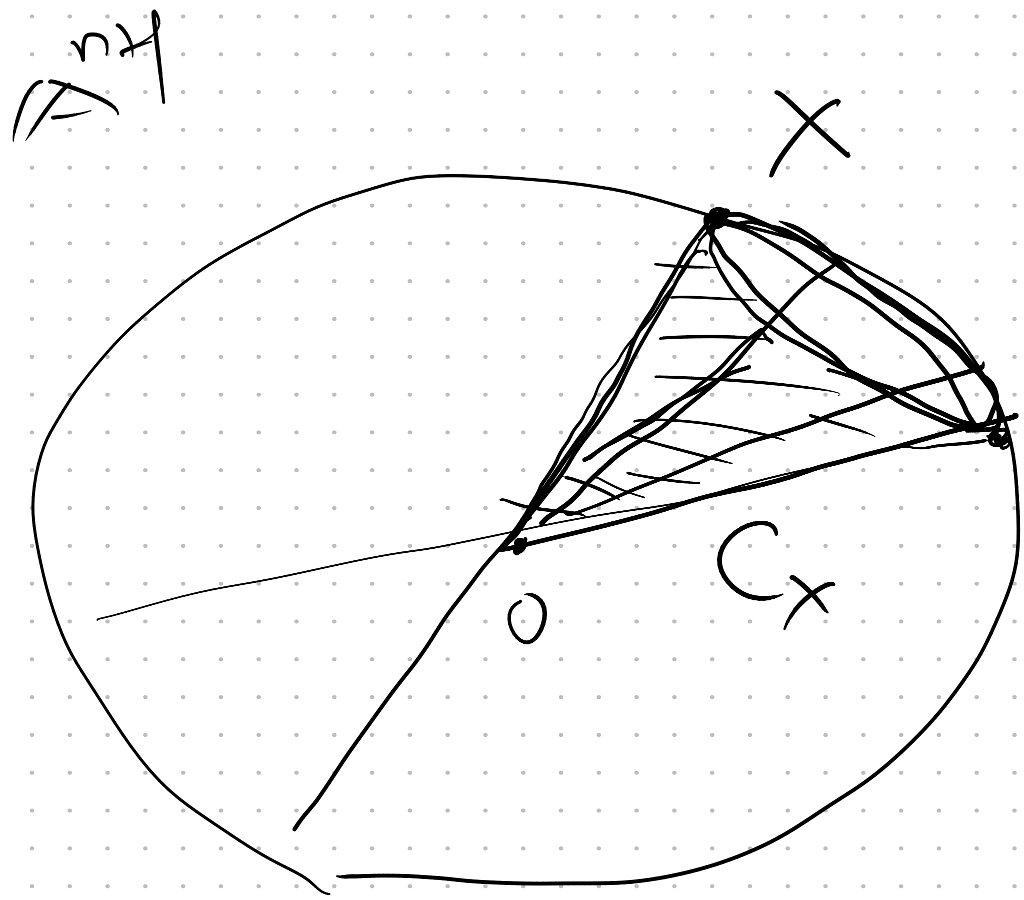
② $X \subset \mathbb{P}^n$ equidim of codim 1.

$C_X \subset \mathbb{A}^{n+1}$

$\parallel$

Closure of $\pi^{-1}(X)$

$\pi : \mathbb{A}^{n+1} \setminus 0 \to \mathbb{P}^n$



Consider

$$\underline{\underline{\mathbb{A}^{n+1} \setminus 0}} \xrightarrow{\pi} \underline{\underline{\mathbb{P}^n}}$$

fibers are $\cong$ $\underline{\underline{\mathbb{A}^1 \setminus 0}}$

Not a product

$$\mathbb{P}^n \times (\mathbb{A}^1 \setminus 0).$$

It is locally a product.

$\exists$ open cover of $\mathbb{P}^n$ over which this is a product:

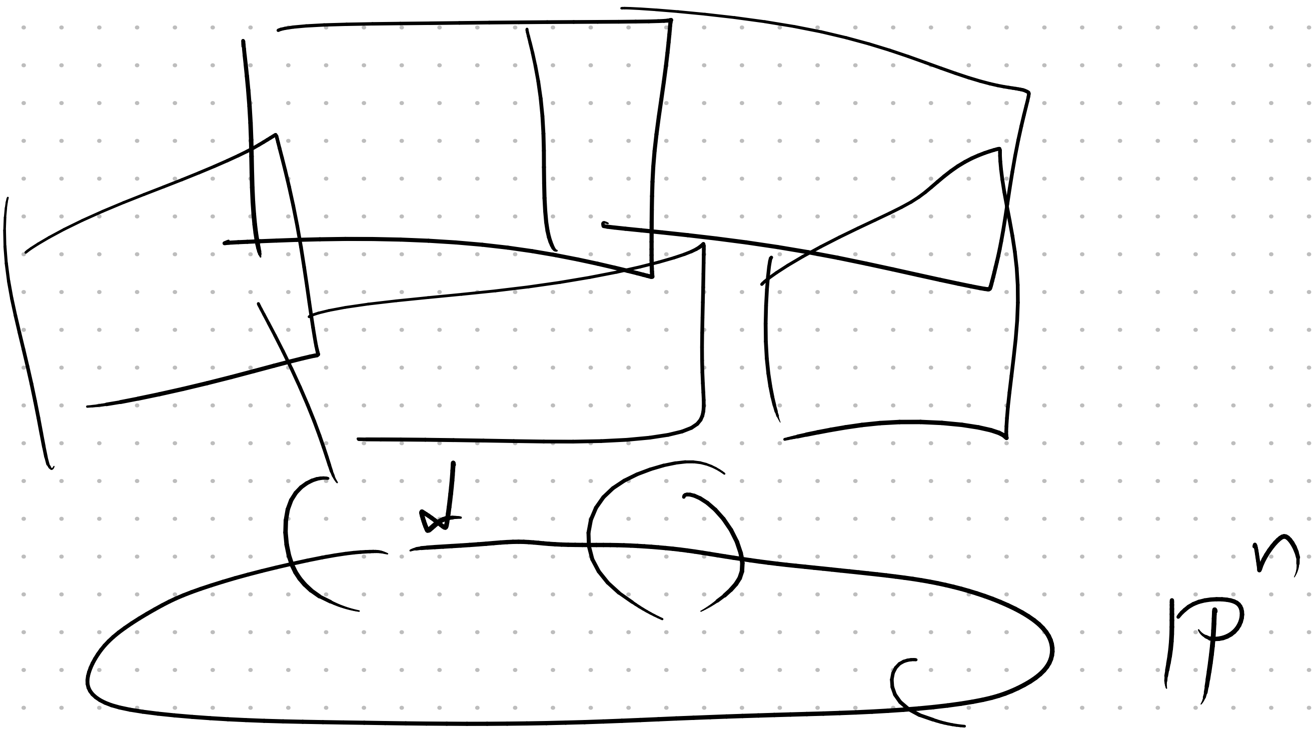
<u>standard</u> $[x_0 : \cdots : x_n]$

$x_0 \neq 0$ U_0

$$\pi^{-1}(U_0) \cong U_0 \times (\mathbb{A}^1 \setminus 0)$$

$$\pi^{-1}(U_0) \cong U_0 \times (\mathbb{A}^1 \setminus 0)$$

$$\begin{array}{ccc} \pi^{-1}(U_0) & \cong & U_0 \times (\mathbb{A}^1 \setminus 0) \\ \rotatebox{90}{$\in$} & & \uparrow \\ (x_0, \dots, x_n) & \longmapsto & ([x_0 : \dots : x_n], x_0) \\ x_0 \neq 0 & & \end{array}$$



Locally trivial fibration

$\underline{\dim_x X}$ only depends locally on X near x.

$$C_X \setminus 0 \longrightarrow X$$

is locally a product with $\mathbb{A}^1 \setminus 0$.

$C_X \setminus 0$ is of dim $\dim X + 1$.

C_X is of dim $\underline{\underline{\dim X + 1}}$.

(X equidim)

$X \subset \mathbb{P}^n$ eq. dim of dim $n-1$

$\rightsquigarrow$

$C_X \subset \mathbb{A}^{\underline{\underline{n+1}}}$ eq. dim of dim $\underline{\underline{n}}$.

$C_X = V(F)$ ← affine $\qquad I(C_X) = (F)$

↳ conical (stable under scaling)

$\Rightarrow$ F has to be homog.

$X = V(F) \subset \mathbb{P}^n$

(3) $X \subset \mathbb{P}^n$ dim r

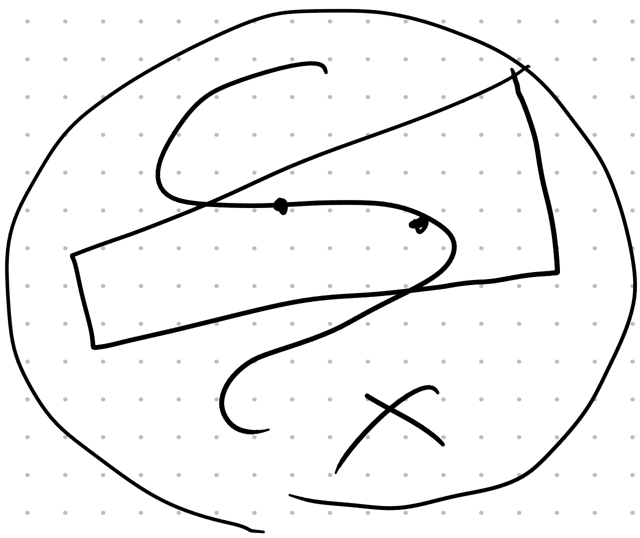
$X_F = X \cap V(F)$

F homog of pos. deg.

$\dim(X_F) \geq r-1$ (P.I.T)

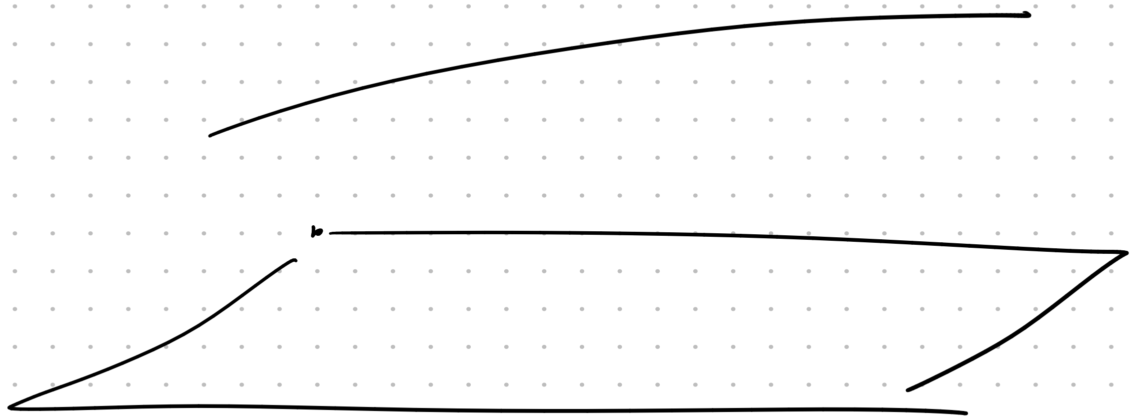
$\underline{\underline{X_F}}$ is <u>non-empty</u> if $r \geq 1$.

"intersection p'ts cannot escape"



contrast with affine spaces

$\mathbb{A}^3$



Consider

$$C_X \subset \mathbb{A}^{n+1}$$

$$C_X \cap V(F) \subset \mathbb{A}^{n+1}$$

$$\dim_0 C_X = r+1 . \quad V(F) \ni 0$$

$$0 \in C_X \cap V(F)$$

↪ why bother?

$\dim_0 (\underbrace{C_x \cap V(F)}) \rightarrow$ Not just $\underline{\underline{0}}$

$\geq \dim C_x - 1$

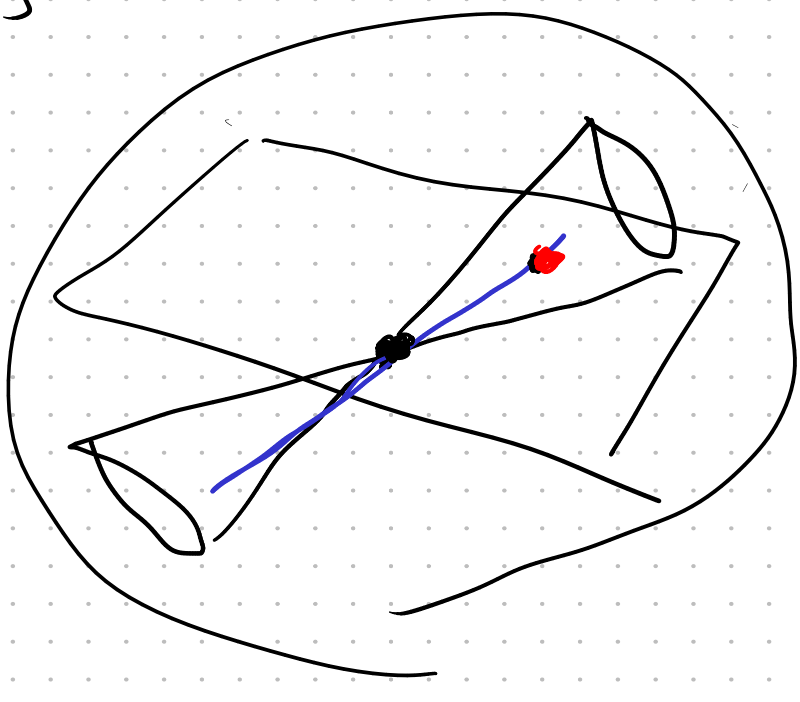
$= (r+1) - 1$

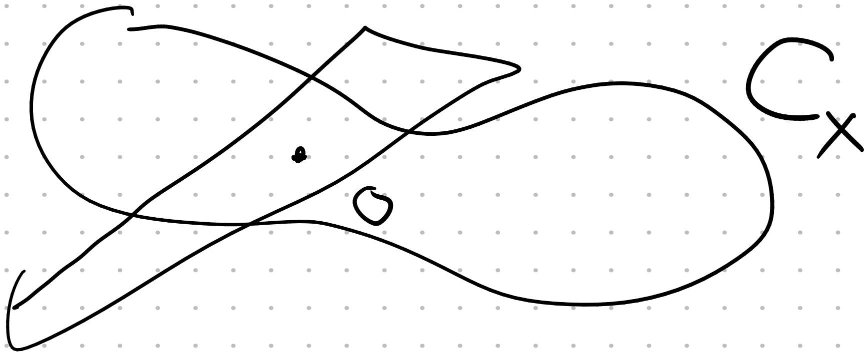
$= \underline{\underline{r}}.$

Know $\underline{\underline{r \geq 1}}$

$\Rightarrow \exists\ p \in C_x \cap V(F)$ $\underline{\underline{p \neq 0}}.$

$\Rightarrow [\underline{\underline{p}}] \in X_F.$





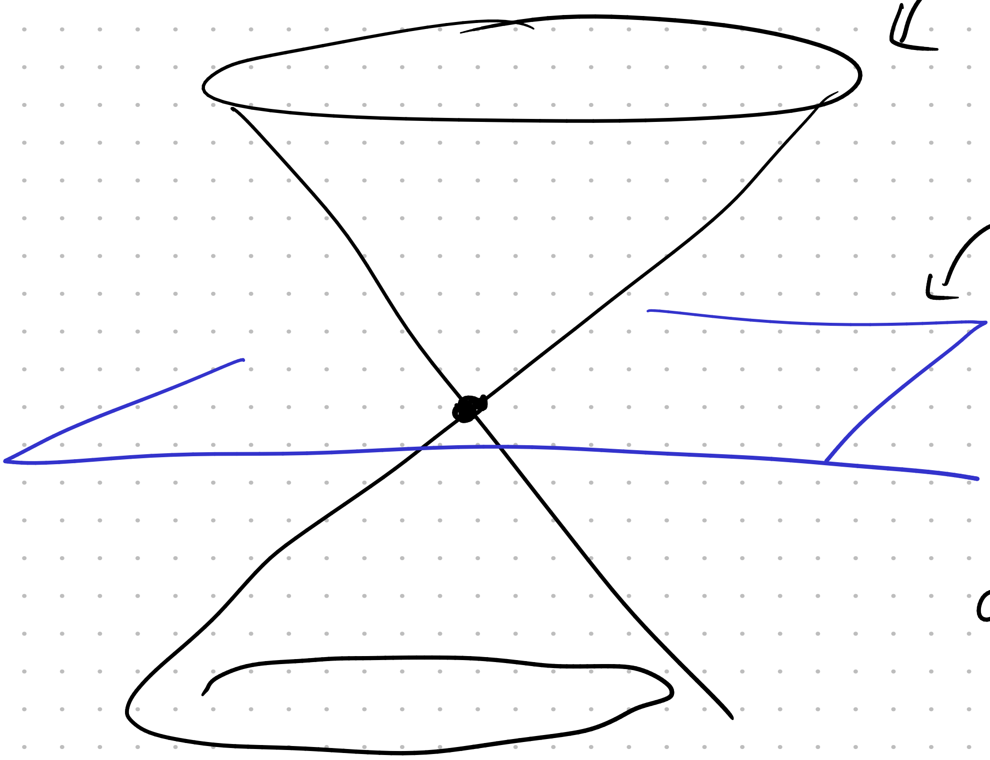
C_x

V(F) on C_x

$$\dim_0 \left(C_x \cap V(F)\right) \geq \dim C_x - 1$$

Real Picture

$\mathbb{R}$-2 dim



one equation.

dim goes down by 2!

BAD

basically x^2+y^2 on $\mathbb{R}^2$

$V(x^2+y^2) = \{0\}$ cut dim down by 2!

one function!

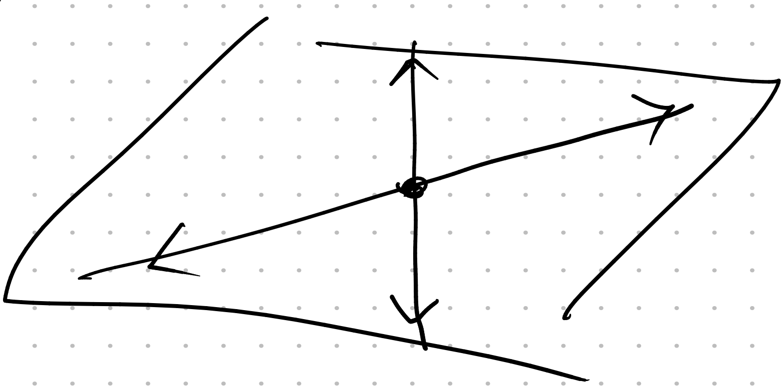
Violation of P.I.T.

$V(x^2+y^2) \subset \mathbb{C}^2$ ← 2 $\dim_{\mathbb{C}}$

$\parallel$

$V(x+iy) \cup V(x-iy)$

1-dim 1-dim



$\mathbb{C}^2$

Cor: $F_1, \dots, F_r$ homog pos deg.

$\hookrightarrow k[x_0, \dots, x_n]$

$r \leq n$

$\mathbb{P}^n \supseteq V(F_1, \dots, F_r)$

Non-empty!

$(\dim \geq n-r)$

Two curves in $\mathbb{P}^2$ must

$\quad = V(F)$

intersect.

$V(F,G)$

$\boxed{\text{Cor: No maps } \mathbb{P}^n \to \mathbb{P}^m \text{ if } n > m.}$ → except constants.

$\mathbb{P}^2 \not\cong \mathbb{P}^1 \times \mathbb{P}^1$

Pf: $U \subset \mathbb{P}^n$ open

$\varphi : U \longrightarrow \mathbb{P}^m$ regular.

Then $\varphi = [F_0 : \cdots : F_m]$

where F_i are hom poly in $k[X_0, \ldots, X_n]$ of same degree.

$X \subset \mathbb{P}^n$ <u>closed</u>.

$\downarrow \longleftarrow$ No single exp. in terms

$\mathbb{P}^r$ of homog. poly.

F_i have no common zeros on

U & $\forall\, u \in U$

$\varphi(u) = [F_0(u) : \cdots : F_m(u)]$.

$\varphi : \mathbb{P}^n \to \mathbb{P}^m$ must have a

global exp.

$\varphi = [F_0 : \cdots : F_m]$

$\hookrightarrow$ F_i have no common zeros

in $\mathbb{P}^n$.

Cannot happen if $m < n$.

e.g. $\mathbb{P}^2 \longrightarrow \mathbb{P}^1$

has to be

$[F : G]$

breaks down at $V(F,G)$.
$\neq \emptyset$

Dim Count:

$X_r \subset \mathbb{A}^{n \times n}$ Rank $\leq r$ matrices

1

$$P \subset \mathbb{A}^{n\times n} \times Gr(n-r, n)$$
$$\| $$
$$\{(M,V) \mid M|_V \equiv 0\}$$

Closed? Check on charts.

$$(*) \quad \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ v_{11} & \cdots & \\ \vdots & & \\ \cdots & & v_{n-r,r} \end{bmatrix}$$

Chart of Gr.
Subsp = col. span

$$\left\{(M,V) \;\middle|\; \underbrace{M|_V = 0}_{M \cdot \text{col. of } (*) \text{ is } 0}\right\}$$

$$M\begin{bmatrix} \vdots \end{bmatrix} = 0 \leftarrow$$ poly equations in entries of M & of column.

$\Longrightarrow$ P is closed.

⑤ (Assume P irred.)

$$\begin{tikzcd} P \arrow[rr, phantom, "\supset"] \arrow[dr] & & P_V \arrow[d, mapsto] \\ & \mathrm{Gr}(n-r,n) \ni & V \end{tikzcd}$$

fiber

$$\underline{\underline{P_v}} \subset \mathbb{A}^{n\times n}$$

$$\begin{array}{c} P_v \\ \| \\ \{M \mid M\underline{\underline{v}} = 0\} \cong \mathbb{A}^{rn}. \end{array}$$

v fixed.

$$\begin{tikzcd} \underline{\underline{M}} : K^n \arrow[r] & K^n \\ V \arrow[u, phantom, "\cup"] \arrow[ur, mapsto] & 0 \arrow[u, phantom] \end{tikzcd}$$

$\Updownarrow$ $\quad V \leftarrow \dim(n-r)$.

$$\overline{M} : \underline{\underline{K^n/V}} \to \underline{\underline{K^n}}$$

$$\begin{array}{c} K^n/V \\ \cong \\ \underline{\underline{K^r}} \end{array} \nearrow \leftarrow r \times n \text{ matrices}$$

Or choose a basis of V

$v_1, \dots, v_{n-r}$, extend to K^n

$$0 \xleftarrow{M} v_1, \dots, v_{n-r},\ (w_1, \dots, w_r) \xrightarrow[M]{} K^n \ \underline{\underline{\text{free}}}$$

$$\begin{tikzcd} \text{irr } \underline{X} \arrow[d] & & \arrow[d] & \text{fiber dim } \underline{c} \\ \text{irr } \underline{Y} & \supset & \underline{U} & \\ & \text{open.} & & \end{tikzcd}$$

Then $\dim X = \dim Y + \underline{c}$

(Because thm on fiber dim) ||

$\exists$ open $U' \subset Y$ over which fiber dim is $\dim X - \dim Y$.

non empty

$\underline{U' \cap U} \Leftarrow$ dim fiber c $=$ $\dim X - \dim Y$

$\dim P$ ✓ $= \underline{\underline{\dim Gr + nr}}$

$$\begin{tikzcd} & P \arrow[dl, "\pi"'] \arrow[dr] & \\ X_r & & Gr \end{tikzcd}$$

fiber dim $\underline{\underline{nr}}$.

$\underline{\underline{\mathrm{Im}}}\,\pi = \underline{\underline{X_r}}$

$M : k^n \to k^n$ $\quad$ 0 on $(n-r)$ dim sub

$\Rightarrow \operatorname{rk} M \leq r.$

$X_r \supset U_r =$ matrices of rk exactly $\underline{\underline{r}}$ ||

open

Over U_r, fibers of π are <u>single</u> pts!

$M : K^n \to K^n \in \underline{U_r}$

$rk(M) = r$

$\ker(M)$ dim $= \underline{(n-r)}$

(M, V) s.t. $M|_V = 0$

$V = \ker M$.

$\Rightarrow$ <u>unique</u> V.

fiber dim 0

want

P

easy to study.

$U_r \subset X_r$

Gr

$\Rightarrow \underbrace{\dim P} = \underbrace{\dim X_r} + 0$

✓

dim counts.