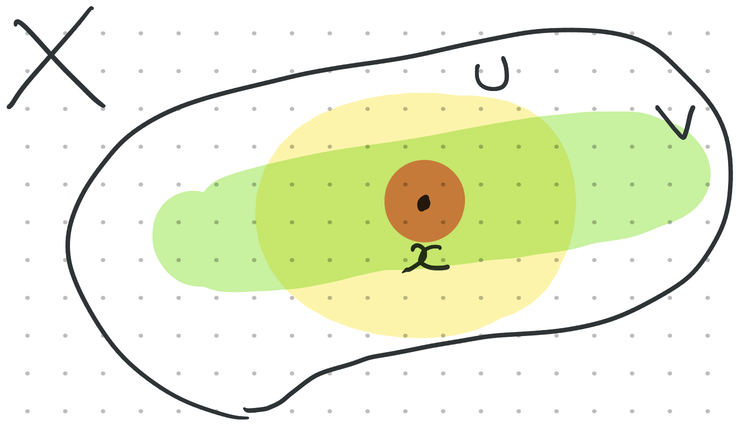
Local Rings / Rings of germs

X alg var $\quad x \in X$



Germ of a regular fun. at x

is an eqv. class of (U, f)

where $U \subset X$ open containing x

& f a reg. fun. on U.

$(U,f) \sim (V,g)$ if $\exists$ open $W \ni x$

s.t. $f|_W = g|_W$

Only the behaviour near x matters.

$\mathcal{O}_{X,x}$ = Set of germs of reg. fun at x.

Examples:

$X = \mathbb{A}^n$, $x = (0, \dots, 0)$

Any polynomial $f \in k[X_1, \dots, X_n]$ defines a germ. (drop U)

$$\frac{x_1 + x_2}{1 + x_3}$$ ← defined in an open $\ni (0, \dots, 0)$

generally $\dfrac{f(X_1, \dots, X_n)}{g(X_1, \dots, X_n)}$ $g(0, \dots, 0) \neq 0$

defines a germ at $(0, \dots, 0)$

$$\mathcal{O}_{\mathbb{A}^n,0} = \left\{ \frac{f}{g} \;\middle|\; \begin{array}{l} f,g \in k[x_1,\dots,x_n] \\ g(0,\dots,0) \neq 0 \end{array} \right\}$$

$$\mathcal{O}_{\mathbb{A}^n,x} \qquad x = (a_1,\dots,a_n)$$

$$\| $$

$$\left\{ \frac{f}{g} \;\middle|\; g(a_1,\dots,a_n) \neq 0 \right\}.$$

$\mathcal{O}_{X,x}$ is a ring.

$\cup$

k as constants.

$\Big\}$ $\underline{k\text{-algebra}}$

$U \subset X$ open

$\ni$

x

$$\mathcal{O}_{X,x} = \mathcal{O}_{U,x}$$

$$\mathcal{O}_{\mathbb{P}^1,[0:1]} = \mathcal{O}_{\mathbb{A}^1,0}.$$

- ($\underline{\underline{\mathcal{O}}}_{X,x}$ is rich even if X is not affine.)

- For computing $\mathcal{O}_{X,x}$, can always reduce to X affine (choose an affine neighborhood).

- $$\begin{array}{rccc} f: & X & \longrightarrow & Y \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & x & \longmapsto & y \end{array}$$

Pullback:

$$f^*: \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$$

<u>Example</u>: $X \subset \mathbb{A}^n$ closed.

<u>Germs</u> of rg. fun at $x \in X$.

$\frac{f}{g}$ f & g are poly.

$g(a) \neq 0$.

$I(X) = \langle f_1, \dots, f_r \rangle$, say.

Just as

$$k[X] = k[\mathbb{A}^n] / \langle f_1, \dots, f_r \rangle$$

$$\mathcal{O}_{X,x} = \mathcal{O}_{\mathbb{A}^n, x} / \langle f_1, \dots, f_r \rangle.$$

Ex $X = V(xy)$.
$x = (0,0)$.

$$\mathcal{O}_{X,x} = \mathcal{O}_{\mathbb{A}^2,0} / (xy)$$

$$= \left\{ \frac{f}{g} \mid g(0,0) \neq 0 \right\} / \langle xy \rangle.$$

$\mathcal{O}_{X,x}$ is a k-algebra.

$$\begin{array}{c} \mathcal{O}_{X,x} \\ \cup \\ m \end{array} = \{ f \in \mathcal{O}_{X,x} \mid f(x) = 0 \}.$$

$$\underline{\underline{m}} \longrightarrow \mathcal{O}_{X,x} \xrightarrow[\text{at } x]{\text{eval}} \underline{\underline{k}}$$

Prop: $\mathcal{O}_{X,x}$ has a unique max ideal m.

Pf: Every non-unit ideal is contained in $\underline{\underline{m}}$.

$\iff$ Every non-unit is contained in m.

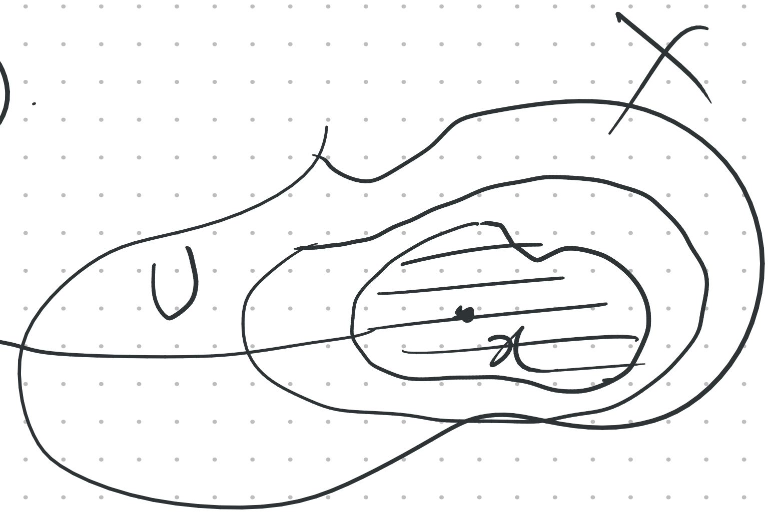
$\iff$ Everything not in m is a unit.

$\mathcal{O}_{X,x} \ni f$ not in m

$f(x) \neq 0$

f non-zero

$1/f$ reg. on



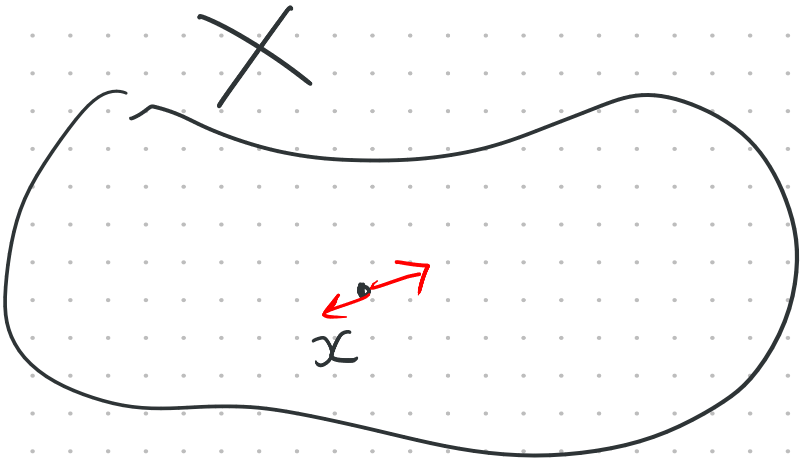
$\Rightarrow f$ is invertible in $\mathcal{O}_{X,x}$.

$\square$

A ring with a unique max id is called a <u>Local ring</u>.

So <u>$\mathcal{O}_{X,x}$</u> is a <u>local</u> ring.

Tangent spaces



Goal: $T_x X$

$\Vert$

Space of tangent vectors to X at x.

① A tangent vect. to X at x is a k-alg. hom

$$\mathcal{O}_{X,x} \longrightarrow k[\varepsilon]/\varepsilon^2 \quad (*)$$

"infinitesimal curve on X centered at x"

⊛ $X \subset \mathbb{A}^n$ closed

$I(X) = \langle f_1, \dots, f_r \rangle$.

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[d] & \supset & k[X] \arrow[dll] \\ k[\varepsilon]/\varepsilon^2 & & \end{tikzcd}$$

$$k[X] \longrightarrow k[\varepsilon]/\varepsilon^2$$

$$\begin{array}{c} k[X] \\ \parallel \\ \dfrac{k[x_1, \dots, x_n]}{(f_1, \dots, f_r)} \longrightarrow k[\varepsilon]/\varepsilon^2 \end{array}$$

$=$

$$\frac{k[x_1,\dots,x_n]}{\langle f_1,\dots,f_r\rangle} \longrightarrow k$$

$$x_i \longmapsto a_i$$

$$p(x_1,\dots,x_n) \longmapsto p(\underline{a_1,\dots,a_n})$$

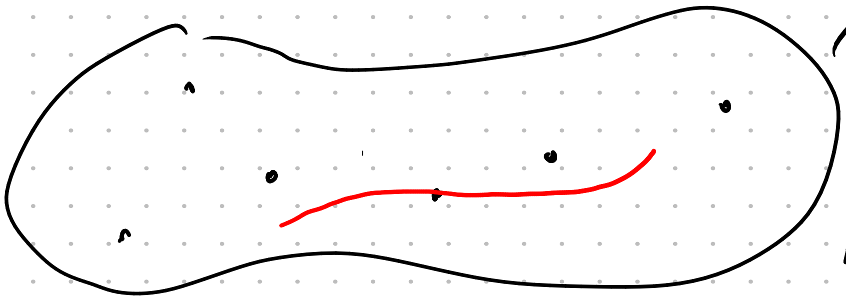
$$f_1,\dots,f_r \longmapsto 0$$

i.e. $\underline{f_i(a_1,\dots,a_n) = 0}$.

<u>Homs</u> $\quad \frac{k[x_1,\dots,x_n]}{\langle f_i\rangle} \longrightarrow k$

$\updownarrow$

Solutions to $\quad f_1 = 0$
(<u>k-valued.</u> $\quad \vdots$
$\qquad\qquad\qquad f_r = 0$



$X \subset \mathbb{A}^n$

$\underline{\text{Homs}}$ $k[X] \to k$ $\leftarrow$ "point in X"

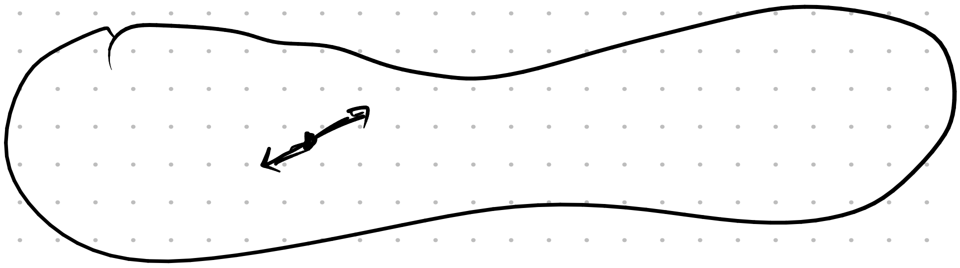
$\underline{\text{Homs}}$ $k[X] \to k[t]$ "curve in X"

$\Updownarrow$

$\underline{\underline{k[t]\text{-valued}}}$ sol^{n}s to

$$\left.\begin{array}{l} f_1 = 0 \\ \vdots \\ f_r = 0 \end{array}\right| \qquad x_i \mapsto a_i(t) \in k[t]$$

$\to$ one-param family (t) of k-valued sol^{n}s.



$a + b\varepsilon$.

$k[x] \longrightarrow k[\varepsilon]/\varepsilon^2$

$k[\varepsilon]/\varepsilon^2$ - valued solutions.

"Truncated polynomial"

"infinitesimal curve"

Tangent vector is 1

$$\mathcal{O}_{X,x} \longrightarrow k[\varepsilon]/\varepsilon^2$$

"inf curve on X centered at x".

$$\begin{tikzcd}
\mathcal{O}_{X,x} \arrow[r, "\varphi"] \arrow[dr] & k[\varepsilon]/\varepsilon^2 \arrow[d, equal] \\
& k \oplus \varepsilon \cdot k \arrow[d] \\
& k
\end{tikzcd}$$

Unique.

evaluation at x.

$$\varphi(f) = f(x) + \varepsilon \cdot \delta(f)$$

$$\delta : \mathcal{O}_{X,x} \longrightarrow k$$

δ is k-linear.

a) $\varphi(f_1 + f_2) = \varphi(f_1) + \varphi(f_2)$.

b) $\varphi(f_1 f_2) = \varphi(f_1)\varphi(f_2)$.

c) $\varphi(c) = c \qquad c \in k$

Translate in terms of δ

(a) $\delta(f_1+f_2) = \delta(f_1)+\delta(f_2)$

(b)

$$f_1f_2(x) + \varepsilon\,\underbrace{\delta(f_1f_2)}$$

$$= \left(f_1(x) + \varepsilon\,\delta(f_1)\right)\left(f_2(x) + \varepsilon\,\delta(f_2)\right)$$

$$= f_1f_2(x) + \varepsilon\cdot\left(f_1(x)\,\delta(f_2) + f_2(x)\,\delta(f_1)\right)$$

$$\delta(f_1f_2) = f_1(x)\,\delta(f_2) + f_2(x)\,\delta(f_1)$$

(c) $\delta(c) = 0 \quad \forall\, c \in k$.

$\delta : \mathcal{O}_{X,x} \to k$ satisfying these are called derivations/k

Homs $\mathcal{O}_{X,x} \to k[\varepsilon]/\varepsilon^2 \Leftarrow$

$\Updownarrow \varphi$

Derivations/k.

$$\delta : \mathcal{O}_{X,x} \longrightarrow k$$

$$\varphi(f) = f(x) + \varepsilon \cdot \delta(f)$$

$$T_x X = \left\{ \begin{array}{c} k\text{-derivations} \\ \mathcal{O}_{X,x} \to k \end{array} \right\}$$

$\hookrightarrow$ k-vector space

Derivations form a k-vspace.

Geometrically

curve ⇝ derivation

infinitesimal ⇝ directional derivative.

Last formulation.

Derivation

$$
\begin{array}{ccl}
\mathcal{O}_{X,a} & \longrightarrow & k \\
\cup & & \\
\mathfrak{m} & \longrightarrow & k \\
\cup & & \\
\mathfrak{m}^2 & \longmapsto & 0
\end{array}
$$

$$\delta(\underline{\underline{f}}\,\underline{\underline{g}}) = \underbrace{f(x)}_{0}\,\delta(g) + \underbrace{g(x)}_{0}\,\delta(f)$$

Derivation $\quad$ k-v.space

$\rightsquigarrow \quad m/m^2 \longrightarrow k$

k-linear map.

Conversely any k-lin. map

$m/m^2 \longrightarrow k$ gives a derivation

$$T_x X \cong \text{Inf. curves at } x$$

$$\cong \text{Der at } x$$

$$\cong \mathrm{Hom}_k(m/m^2, k)$$

<u>Ex.</u> $\mathbb{A}^2$. $x = (0,0)$.

$$\mathcal{O}_{\mathbb{A}^2,0}$$
$$\cup$$
$$\mathfrak{m} = \langle x, y \rangle$$
$$\cup$$
$$\mathfrak{m}^2 = \langle x^2, xy, y^2 \rangle.$$

$$\mathfrak{m}/\mathfrak{m}^2 = k\langle x \rangle \oplus k\langle y \rangle \leftarrow$$
$\hookrightarrow$ 2 dim

$$T_{\mathbb{A}^2,0} \cong \mathrm{Hom}(k^2, k)$$
$$\cong k^2$$

$$X = V(x+y^2) \subset \mathbb{A}^2$$

$$\mathcal{O}_{X,x} = \mathcal{O}_{\mathbb{A}^2,0} \Big/ (x+y^2)$$

$$\mathfrak{m}/\mathfrak{m}^2 = \underline{\underline{\left(\mathfrak{m}/\mathfrak{m}^2\right)}} \Big/ (x+y^2)$$

$$= \underline{\underline{\langle x, y \rangle}} \Big/ \underline{\underline{(x+y^2)}}$$

only <u>linear</u> terms matter
(higher degree terms $\in \mathfrak{m}^2$)

$\mathfrak{m}/\mathfrak{m}^2$ for $\mathcal{O}_{X,x} \cong \underline{\underline{\langle y \rangle}}$.

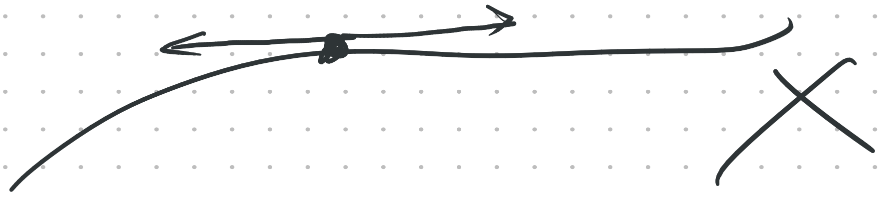
Dual is also one-dim.

Fundamental inequality

$$\dim T_x X \geq \dim_x X$$

Def. If equality holds, say that X is **smooth** or **non-sing** at x.

smooth



singular