Germs, local rings, tangents

(1) $X \subset \mathbb{A}^n$ closed

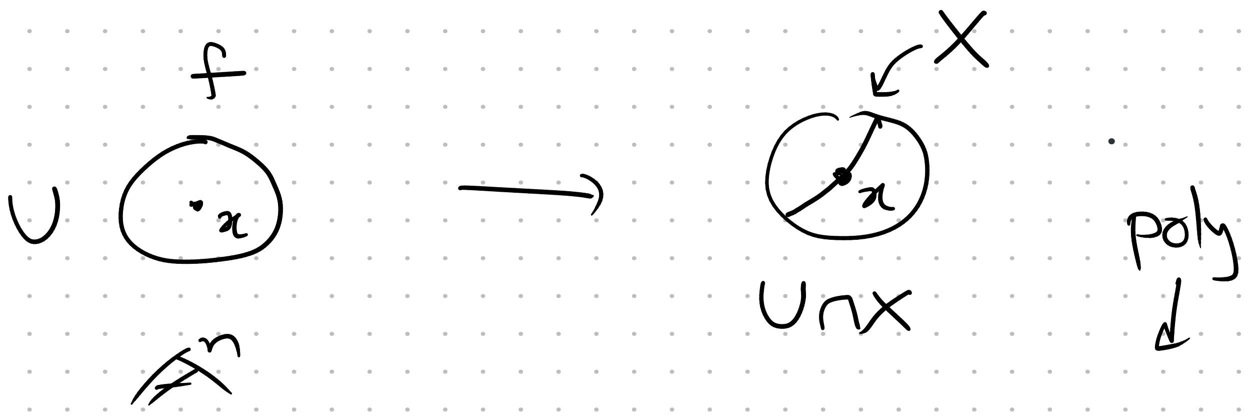
$I(X) = \langle f_1, \dots, f_r \rangle \subset k[x_1, \dots, x_n]$

$$\mathcal{O}_{X,x} \cong \mathcal{O}_{\mathbb{A}^n, x} / \langle f_1, \dots, f_r \rangle$$

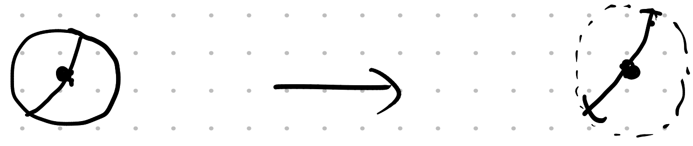
i.e.

$$\mathcal{O}_{\mathbb{A}^n, x} \xrightarrow{\text{restrict}} \mathcal{O}_{X, x}$$

is surj & kernel is gen by $f_1, \dots, f_r$.



surj



$\bar{f} = \frac{p}{q}$

$q(x) \neq 0$

$\frac{p}{q}$

Suppose $f \in \mathcal{O}_{\mathbb{A}^n, x}$ goes to 0.

$f \sim (U, f)$ $\quad U \subset \mathbb{A}^n \quad x \in U$.

$(U \cap X, f) \sim 0$ germ.

After possibly shrinking U, we have

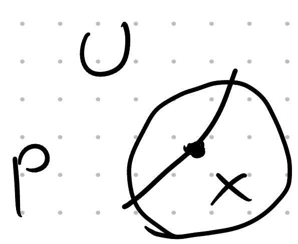
$f|_{U \cap X} = 0$.

After possibly shrinking U,

$f = \frac{p}{q}$ ← polynomials. $q \neq 0$ on U.

so $p|_{U \cap X} = 0$

But don't know $p = 0$ on all of X.



← complement of U
$= V(g_i \cdots)$
$g(x) \neq 0$

$p(x) \cdot g(x) = 0$
on all of X

Can write

$$p \cdot g = \sum a_i \cdot f_i \in k[x_1, \dots, x_n]$$

Divide by g $\leftarrow$ justify.

$g(x) \neq 0 \Rightarrow g$ is invertibl in an open set containing x.

$\Rightarrow g$ is invertible in $\mathcal{O}_{\mathbb{A}^n, x}$.

$$p = \sum \frac{a_i}{g} \cdot f_i$$

$$\underset{\uparrow}{\frac{p}{q}} = \sum \underbrace{\frac{a_i}{gq}} \cdot \underset{\uparrow}{f_i}$$

<u>wanted</u>.

some other <u>germs</u>. $\in \mathcal{O}_{\mathbb{A}^n, x}$.

- Tangent spaces.

$x \in X$.

① Ring hom $\mathcal{O}_{X,x} \to k[\varepsilon]/\varepsilon^2$

② k-Derivations

$$\mathcal{O}_{X,x} \to k$$

Opaque? The def. is indep of coordinates!

Also give functoriality.

$$f : X \to Y$$
$$x \mapsto y \quad (x \in X,\ y \in Y)$$

induces:

$$df : T_{X,x} \to T_{Y,y}.$$

$$\begin{tikzcd} f^*: \mathcal{O}_{Y,y} \arrow[r] \arrow[dr, "v \circ f^*"'] & \mathcal{O}_{X,x} \arrow[d, dashed, "v"] \\ & k[\varepsilon]/\varepsilon^2 \end{tikzcd}$$

Ring Hom $\mathcal{O}_{X,x} \to k[\varepsilon]/\varepsilon^2$

v

$\downarrow$

Ring hom $\mathcal{O}_{Y,y} \to k[\varepsilon]/\varepsilon^2$

$v \circ f^*$

③ $X \subset \mathbb{A}^n$ $\quad I(X) = \langle f_1, \dots, f_r \rangle$

$x = (a_1, \dots, a_n)$

① $\Longleftrightarrow$ $k[\varepsilon]/\varepsilon^2$ - valued sol^{n}s to $f_1 = 0$

$\vdots$

$f_r = 0$

$\mathcal{O}_{X,x} \xrightarrow{\varphi} k[\varepsilon]/\varepsilon^2 \to$ Sol^n.

$x_i \mapsto (a_i + b_i\varepsilon)$

$K[\varepsilon]/\varepsilon^2$ - valued sol^{n}s means

$(a_1 + b_1\varepsilon, \dots, a_n + b_n\varepsilon)$

$$f_i(a_1 + b_1\varepsilon, \dots, a_n + b_n\varepsilon) \in k[\varepsilon]/\varepsilon^2$$
$$\parallel$$
$$0 = 0 + 0 \cdot \varepsilon.$$

$$\mathcal{O}_{X,x} = \mathcal{O}_{\mathbb{A}^n,x} / \langle f_1, \dots, f_r \rangle$$

f_i represents 0 of $\mathcal{O}_{X,x}$.

$$\underset{=}{f_i} \xmapsto{\varphi} f_i(a_1 + b_1\varepsilon, \dots, a_n + b_n\varepsilon) = 0$$

To go back. $(a_i + b_i\varepsilon)$ a sol^n.

$X_i \mapsto a_i + b_i\varepsilon$

$\mathcal{O}_{\mathbb{A}^n, x} \longrightarrow k[\varepsilon]/\varepsilon^2$ &

show that it sends $f_i \mapsto 0$ so it descends to the quotient. ✓

$$(U, f) = \left(U, \frac{p}{q}\right) \quad q(x) \neq 0.$$

$$\longmapsto \frac{p(a_i + b_i \varepsilon)}{q(a_i + b_i \varepsilon)}$$

makes sense in $k[\varepsilon]/\varepsilon^2$?

justify the division.

$$q(a_i + b_i\varepsilon) = \underset{\substack{\uparrow \\ \text{non-zero.}}}{\underset{q(a_1, \dots, a_n)}{\underline{\qquad}}} + \underline{\qquad} \varepsilon$$

$a + b\varepsilon \quad a \neq 0 \quad$ is invertible

$$(a + b\varepsilon)^{-1} = \left(\frac{1}{a} - \dots \varepsilon\right)$$

$(U, P/q) \sim (U', P'/q')$

then $\frac{P}{q} \overset{?}{=} p'/q'$

$\exists$ open $V \subset U \cap U'$ where

$P/q = P'/q'$ i.e. $pq' = p'q$ ← as poly.

then it follows that

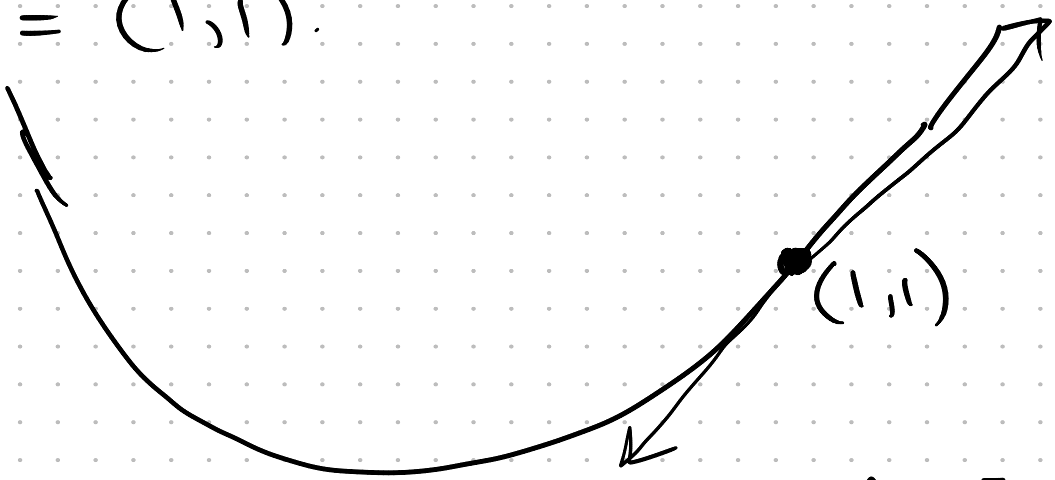
$p(a_i + h_i\varepsilon)\, q'(a_i + h_i\varepsilon) = p' \ldots q \ldots$

$\frac{P}{q}(a_i + h_i\varepsilon) = \frac{P'}{q'}(a_i + h_i\varepsilon).$

$f_i \mapsto 0$ because we have a solution.

Example: $f(x,y) = y - x^2$

$a = (1,1)$.



$(1,1) + \varepsilon(b_1,b_2)$ $\qquad$ $\varepsilon \in \underline{\underline{k[\varepsilon]}}/\underline{\underline{\varepsilon^2}}$

$(1+\varepsilon b_1, 1+\varepsilon b_2)$

When does this lie on the curve?

$$(1+\varepsilon b_2) - (1+\varepsilon b_1)^2 = 0$$

$$\underbrace{1+\varepsilon b_2 - 1 - 2\varepsilon b_1 - \cancelto{0}{\varepsilon^2 b_1^2}}_{\text{Actual expansion.}}$$

Remains only the ε-term.

$$\varepsilon(b_2 - 2b_1) = 0$$

Conclusion

$(1,1) + \varepsilon(b_1, b_2)$ lies on the curve iff $b_2 - 2b_1 = 0$

(Lying on the subvar will be a system of lin equations in the ε coeff.)

$(a_1, \dots, a_n) \in X$.

$(a_1 + \varepsilon b_1, \dots, a_n + b_n \varepsilon)$ $/\varepsilon^2$

$f_i(\dots) = 0$?

$f_i(a_1, \dots, a_n) = 0$

No const term

No ε^2 term.

Only ε-term. $\leftarrow$ Linear comb of b_i's.

only if def in y coord is twice def in x coord.

(3). A third eqv. formulation of tangent space is:

$\mathrm{Hom}_k(m/m^2, k)$

$X \subset \mathbb{A}^n$ $\quad m = (x_i - a_i) \subset \mathcal{O}_{\mathbb{A}^n, x}$

$m/m^2 = k$. v. space with basis $(x_1 - a_1, \ldots, x_n - a_n)$

$\uparrow$

$\sum c_i(x_i - a_i) \quad c_i \in k$.

For m/m^2 for X, we need to further quotient by $f_1, \ldots, f_r$.

$\overline{f_i}$ = image of f_i in m/m^2

$\bar{f}_i$ = "Linear part of f_i".

$f_i(x_1, \ldots, x_n) = c_0 +$

$\boxed{c_1(x_1 - a_1) + \cdots + c_n(x_n - a_n)}$

$+$ quadratics $+$ cubics $+ \cdots$

$c_0 = f_i(a_1, \ldots, a_n)$ in X.
$= 0$

$c_i = \dfrac{\partial f}{\partial x_i}(a_1, \ldots, a_n)$

formal partial.

$\mathfrak{m}/(\mathfrak{m}^2 + f_1, \ldots, f_r) \leftarrow$

$\wr\wr$

$(\mathfrak{m}/\mathfrak{m}^2)/(\bar{f}_1, \ldots, \bar{f}_r) \leftarrow$

Want to solve

$$f_i(a_1+b_1\varepsilon, \dots, a_n+b_n\varepsilon) = 0$$

$$f_i(a_1,\dots,a_n) = 0 \quad \checkmark$$

$$\parallel$$

$$f_i(a_1,\dots,a_n) + \boxed{\frac{\partial f_i}{\partial x_1}(a_1,\dots,a_n)\, b_1 \varepsilon + \frac{\partial f_i}{\partial x_2}(a_1,\dots,a_n)\, b_2 \varepsilon + \dots + \frac{\partial f_i}{\partial x_n}(a_1,\dots,a_n)\, b_n \varepsilon = 0}$$

in k

(4) Tangent space to

$V(xy-z^2) \subset \mathbb{A}^3$ at

$\langle x,y,z \rangle / xy-z^2$

$(0,0,0)$.

$(0,0,0) + \varepsilon(b_1, b_2, b_3)$

when is this a sol^n to

$$xy - z^2 = 0$$

$$(\varepsilon b_1 \cdot \varepsilon b_2 - (\varepsilon b_3)^2) = 0$$

Always!

T is 3-dimensional consists of all (b_1, b_2, b_3).

$\underline{\underline{(xy - z)}}$

$(0,0,0) + \varepsilon \cancel{(b_1,} b_2, b_3)$

$(\varepsilon \cdot b_1)(\varepsilon b_2) - \varepsilon b_3 = 0$

$\Rightarrow b_3 = 0.$

i.e. ~~t~~ the tangent space is 2-dim

$\underline{\underline{(b_1, b_2, 0)}}.$

Gen. by $(1,0,0)$
$(0,1,0).$

In terms of $Hom(\underline{\underline{m/m^2}}, k)$

in the amb. space

$m/m^2 = \langle x, y, z \rangle$

for X

$$m/m^2 = \langle x, y, z \rangle \Big/ \overline{\underline{xy} - z}$$

$$= \langle x, y, z \rangle / z$$

$$m/m^2 \cong \langle x, y \rangle$$

$\mathrm{Hom}_k(\underline{m/m^2}, k)$ has same dim as m/m^2

$X \mapsto a$
$Y \mapsto b$ $\leftarrow$ fixes the hom.

$$\underline{\underline{\begin{array}{l} X \mapsto 1 \\ Y \mapsto 0 \end{array}}} \quad \& \quad \underline{\underline{\begin{array}{l} X \mapsto 0 \\ Y \mapsto 1 \end{array}}}$$

Form a basis of $\mathrm{Hom}(m/m^2, k)$

(6). $\underset{\text{Char} \neq 3}{V(X^3+Y^3+Z^3)} \subset \mathbb{P}^2$

Smooth at: $[1:-1:0]$

Tangent space is one dim.

move to an affine $X=1$

$V(1+y^3+z^3) \subset \mathbb{A}^2_{(y,z)}$

$(-1,0)$

$$\frac{m}{m^2} = (y+1, z) \Big/ \frac{\cong \boxed{(z)}}{1+y^3+z^3}$$

expand it in $(y-1), z$

$(y+1) \cdot (3y^2)\big|_{-1,0}$

$+$

$z \cdot (3z^2)\big|_{(-1,0)}$

$= \boxed{(y+1) \cdot (+3)} + 0.$