Completeness

Recall $\pi : A \to B$ is closed if the image of every closed sub of A is a closed subset of B.

Non ex: $\mathbb{A}^2 \xrightarrow{\pi} \mathbb{A}^1$ Not closed.

$(x, y) \longmapsto x$

Consider

$$\begin{array}{ccc} V(xy-1) & \subset & \mathbb{A}^2 \quad \text{closed} \\ \downarrow \pi & & \\ \mathbb{A}^1 \setminus \{0\} & \subset & \mathbb{A}^1 \quad \text{not closed.} \end{array}$$

Def: A variety X is complete if for all Y, the projection

$$X \times Y \to Y$$

is closed.

Consequences (Complete $\sim$ compact)

analogous

Prop: $f : X \to Y$, X complete, Y separated.

then $f(X) \subset Y$ is closed.

Pf. Look at $\Gamma_f \subset X \times Y$

$$\Gamma_f = \{(x, f(x))\}$$

$$\begin{array}{ccc} X \times Y & \xrightarrow{(f, id)} & Y \times Y \\ \cup \text{ closed} & & \cup \text{ closed} \\ \Gamma_f & \longrightarrow & \Delta \end{array}$$

$$\begin{array}{cl} \Gamma_f \subset X \times Y & \text{closed} \\ \pi \big\downarrow & \pi(\Gamma_f) = f(X) \text{ is closed} \\ Y & \text{because } \pi \text{ is closed.} \end{array}$$

$\square$

Thm: All projective varieties are complete

Cor: $f : X \to Y$ ← separated
projective

$f(X)$ is closed

Ex. V_d = Homog poly of deg d in X, Y

$\mathbb{P}V_d \supset T$ = poly with a triple zero.
closed.

$$\mathbb{P}V_1 \times \mathbb{P}V_{d-3} \longrightarrow \mathbb{P}V_d$$
$$L, F \mapsto L^3 \cdot F$$

Conseq:

X proj & connected.

Then the only reg fun on X are constants.

Pf: $f \in k[X]$

$$f : X \to \mathbb{A}^1 \subset \mathbb{P}^1$$

$$\bar{f} : X \to \mathbb{P}^1 \quad \leftarrow \text{misses } \infty = [1:0]$$

proj.

$\Rightarrow \bar{f}(X) \subset \mathbb{P}^1$ closed.

$\cancel{\mathbb{P}^1}$ or finite sets.

X connected $\Rightarrow \bar{f}(X)$ connected

so $\bar{f}(X) = \{\cdot\}$

$\Rightarrow f$ is constant. $\square$

Conseq :

V_d = Homog. poly of deg d in X_0, X_1, X_2

$\mathbb{P}V_d \leftarrow$ "plane curves"

$$\mathbb{P}V_d \supseteq \Delta = \left\{ [F] \,\middle|\, \exists\, p \in \mathbb{P}^2 \text{ s.t. } \frac{\partial F}{\partial X_i}(p) = 0 \quad i=0,1,2 \right\}$$

$\hookrightarrow$ closed

Why?

$$\underset{\text{closed}}{Z} \subset \mathbb{P}V_d \times \mathbb{P}^2$$

(F, p)

$$Z = \left\{ (F,p) \,\middle|\, \frac{\partial F}{\partial X_i}(p) = 0 \quad i=0,1,2 \right\}$$

$\hookrightarrow$ poly eq^n in coeff of F & coord of p.

$$Z \xrightarrow{\pi} \Delta$$

$\pi : \mathbb{P}V_d \times \mathbb{P}^2 \longrightarrow \mathbb{P}V_d$

$\hookrightarrow$ closed

complete.

$\square$

Roughly.

$$S = \{ \exists p \; \underbrace{\cdots\cdots}_{\text{closed conditions}} \}$$

closed conditions
↳ poly = 0

$$\exists p \in X$$

X ↳ complete.

then S is closed.

Homework :- $\dim \Delta = \dim \mathbb{P}V_d - 1$

↳ codim 1.

$\Delta = V(\text{one equation})$.

$F = \sum a_I X^I$ homog in X_0, X_1, X_2

Then $\exists$ polynomial in the a_I's whose vanishing

$\Updownarrow$

F being singular.

Ex. $aX^2 + bY^2 + cZ^2 + dXY + eYZ + fXZ = 0$

singulir if

$\_\ \_\ \_\ \_\ \_\ \_\ \_\_\_ = 0$

↳ deg 3 & has 6 terms.

$\underline{\underline{a}}X^3 + \underline{\underline{b}}Y^3 + \underline{\underline{c}}Z^3 + \underline{\underline{d}}X^2Y + \cdots -$

↳ singular ?

↳ Turns out has deg $\underline{\underline{12}}$

↳ 1040 $\underline{\underline{\text{terms}}}$.

<u>Proof</u>: that proj. var are complete.

$X \times Y \to Y$ is closed.

① $X \subset \mathbb{P}^n$ is closed, so

$\underline{\underline{\mathbb{P}^n}} \times Y \to Y$ is closed.

$\Rightarrow X \times Y \to Y$ is closed.

② $X \times Y \to Y$ is closed can be checked on an open cover of Y.

$Y = \bigcup U_i$ &

$X \times U_i \to U_i$ closed $\forall i$

$\Leftrightarrow$ $X \times Y \to Y$ is closed.

Every Y has an affine cover.
so suffices to treat Y affine.

3 Y affine, $Y \subset \mathbb{A}^m$ closed.

$$\begin{array}{ccccc} X \times Y & \to & Y & & \text{closed.} \\ \cap & & \cap & & \Uparrow \\ X \times \mathbb{A}^n & \to & \mathbb{A}^n & & \text{closed} \end{array}$$

$$\boxed{\mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m \text{ is closed}}$$

$$Z \subset \mathbb{P}^n_X \times \underbrace{\mathbb{A}^m}_{(t_1,\dots,t_m)} \quad \text{closed}$$

$$\mathbb{P}^n = [X_0 : \dots : X_n]$$

$$\begin{array}{c} Z \\ \pi \downarrow \\ \mathbb{A}^m \end{array}$$

$$Z = \{ F_i(X_0,\dots,X_n, \underline{\underline{t_1,\dots,t_m}}) = 0 \quad i=1,\dots,r \}$$

Homog in X - variables.

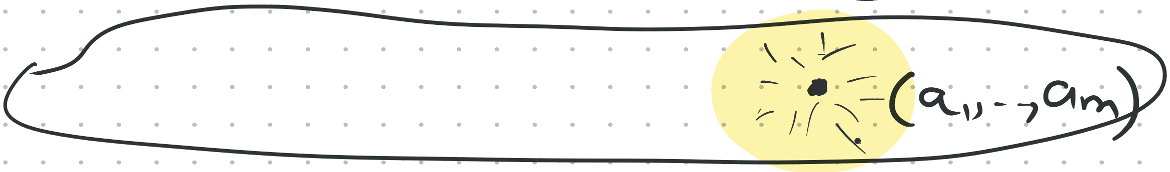
$$\pi(Z) = \left\{ t \in \mathbb{A}^m \;\middle|\; \begin{array}{l} F_i(X,t)=0 \\ \text{has a} \\ \text{non-zero sol}^n \\ \text{in } X \end{array} \right\}$$

Let's show the complement is open.

$t \notin \pi(Z)$ & $t=(a_1,\dots,a_m)$

ie $F_i(x,a)=0$ has no non-zero sol^n.

No sol^n.



Nullstell:

$$\underbrace{\sqrt{\langle F_i(x,a)\rangle}}_{\cup} \subset k[x_0,\dots,x_n]$$

$$(x_0,\dots,x_n) \quad \exists\ N$$

$$\boxed{(x_0,\dots,x_n)^N \subset \langle F_i(x,a)\rangle}$$

mean?

$$X^I = \sum G_i \, F_i(x,a)$$

X^I deg N

G_i: homog of $N-d_i$

F_i: homog deg d_i

Hunt for G_i's.

V_d = Homog poly of deg d in $X_0, \dots, X_n$.

Look at the map for $t \in \mathbb{A}^m$

$$\begin{array}{c} \| \; V_{N-d_1} \oplus V_{N-d_2} \oplus \cdots \oplus V_{N-d_r} \\ \downarrow \\ \| \qquad V_N \end{array}$$

$$(G_1, G_2, \ldots, G_r)$$

$$\Big\downarrow M_t$$

$$G_1(x) \cdot \underline{F_1(X,t)} + \cdots + G_r(x) \underline{F_r(X,t)}$$

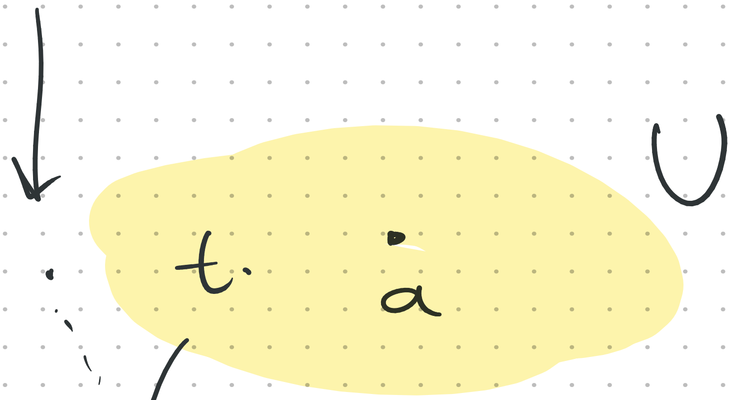
M_t is a k-linear map. bet^n two k-v. spaces

$$M_t = \left.\begin{pmatrix} - & & \\ - & & \\ \vdots & & \\ - & & \end{pmatrix}\right\} c$$

depend on t but entries are <u>poly</u> <u>in t</u>.

then $\forall\ t \in U$ the system

$F_i(x,t)=0$ has no non-zero solutions!



No sol'ns!

We Know

$\underline{\underline{M_a}}$ is $\underline{\underline{\text{surjective}}}$!

$\underline{\underline{c}} = \dim V_N$.

$\exists$ $\underline{\text{non-zero}}$ $\underline{c \times c}$ minor.

in $\underline{\underline{M}}_a$

Let $U \subset \mathbb{A}^m$ be the set

where this $c \times c$ minor is

non-zero.

$\underline{\underline{U \text{ is open}}} \ni \underline{\underline{a}}$

& $\forall\, t \in U$, $\underline{\underline{M_t}}$ is $\underline{\underline{\text{surj}}}$!