Closed conditions

$$Gr(1,4) \times Gr(2,4) \supset \{(L,V) \mid L \subset V\} = Z$$

Z is a closed subset.

↳ Locally, in charts, Z is defined as the vanishing locus of polynomials.

Chart: $Gr(2,4)$

Represent $V = \text{Col. span} \begin{array}{c} \begin{matrix} v & w \end{matrix} \\ \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \\ v_4 & w_4 \end{bmatrix} \end{array}$

& choose a 2-elem subset of $\{1,2,3,4\}$ & make the corresponding submatrix $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ & rest of the coordinates give a chart.

. . .

Similarly $L = \underset{\text{span}}{\text{col}} \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{bmatrix}$

In a chart, we make $l_i = 1$.

In terms of $\underline{v}_i, \underline{w}_j, \underline{l}_k$ how do we express $L \subset V$?

$\rightarrow \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{bmatrix} \in \text{Span}\langle \underline{v}, \underline{w} \rangle$

$\Leftrightarrow \begin{bmatrix} v_1 & w_1 & l_1 \\ v_2 & w_2 & l_2 \\ v_3 & w_3 & l_3 \\ v_4 & w_4 & l_4 \end{bmatrix}$ $\underline{\underline{\text{col}}}$ span of $\underline{\underline{\text{dim } 2}}$

$\Leftrightarrow$ All 3×3 minors are 0.

↑ polynomials in v, w, l.

$\square$

V_d = Homog. poly of deg d in X_0, X_1, X_2, X_3

$\mathbb{P}V_d \times Gr(2,4)$ ↳ space of lines ($\mathbb{P}^1$s) in $\mathbb{P}^3$

$\cup$

$\{([F], L) \mid L \subset V(F)\} = Z$

Z is closed.

↳ In charts defined by polynomials.

$F = \sum_{\deg d} a_I X^I \qquad X^I = X_0^{i_0} X_1^{i_1} X_2^{i_2} X_3^{i_3}$

$[F] = [a_I] \leftarrow$ "Homog coord on $\mathbb{P}V_d$"

In a chart, we make $a_I = 1$ for some I & rest give coordinates.

$$L = \mathbb{P}.\ \mathrm{Span}\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \\ v_4 & w_4 \end{bmatrix}$$

$=$ ↪ Local coordinates after making a sub $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$L \subset V(F)$?

$$[\lambda v_1 + \mu w_1 : \cdots : \lambda v_4 + \mu w_4]$$

$$[\lambda : \mu] \in \mathbb{P}^1$$

$$F(\lambda v_1 + \mu w_1, \dots, \lambda v_4 + \mu w_4) = 0 \quad \forall \lambda, \mu$$

$$= \sum \boxed{a_I} (\lambda v_1 + \mu w_1)^{i_1} \cdots (\lambda_4 v_4 + \mu w_4)^{i_4}$$

$$= \_\_\_\ \lambda^d + \_\_\_\ \lambda^{d-1}\mu + \cdots + \_\_\_\ \mu^d \equiv 0$$

(*) ↪ poly in a_I, v_i, w_j.

$Z =$ Vanishing of all coeff of $\circledast$

poly in $\underline{\underline{a_I}}, \underline{\underline{v_i, w_j}}$

Local coordinates.

$Z \subset \mathbb{P}V_d \times G(2,4)$ $\underline{\text{closed}}$.

$$\begin{array}{c} Z \\ \pi \downarrow \\ G(2,4) \end{array}$$

Fibers of π ?

Fix a line $L \subset \mathbb{P}^3$

$\pi^{-1}(L) = \{ [F] \mid F$ vanishes when restricted to $L \}$

$\underline{\underline{F}}(x_i) \overset{\text{sub}}{\longmapsto} F(\lambda v_i + \mu w_i)$

Linear in F

$F(x_i) + G(x_i) \xmapsto{\text{sub}} F(\lambda v_i + \mu w_i) + G(\lambda v_i + \mu w_i)$

Say $\underline{L} = \{ X_2 = 0 \ \& \ X_3 = 0 \}$

$= \left\{ \begin{bmatrix} x_0 \\ x_1 \\ 0 \\ 0 \end{bmatrix} \middle| \ [x_0 : x_1] \in \mathbb{P}^1 \right\}$

$F(X_0, X_1, X_2, X_3)$ restricts to 0 on L.

↳ i.e. $F(X_0, X_1, 0, 0) = 0$

$\pi^{-1}(L) = \{ [F] \mid F(x_0, x_1, 0, 0) = 0 \}$

all terms of F are div by either X_2 or X_3

↳ NO pure X_0, X_1 terms.

$\parallel$

Span of monomials of deg d

avoiding those that have only X_0, X_1.

$\pi^{-1}(L) = \underline{\mathbb{P}}\langle$ span of monomials except pure x_0, x_1 monomials $\rangle$

$$\begin{tikzcd} Z \arrow[d, "\pi"] & \supset & \mathbb{P}\langle \cdots \rangle \arrow[d] \\ Gr(2,4) & \ni & L \end{tikzcd}$$

← irred.

Let's show: All fibers of π are irred & of same dim.

$\Rightarrow Z$ is irreducible.

What if I choose a different $L \subset \mathbb{P}^3$ i.e. diff. $\underline{2}$ dim $\subset$ $\underline{4}$-dim.

Change your basis to

Y_0, Y_1, Y_2, Y_4

so that 2-dim sub $=$ Span$\langle Y_0, Y_1 \rangle$

Then in this basis

$$L = \left\{ \begin{bmatrix} * \\ * \\ 0 \\ 0 \end{bmatrix} \right\}$$ & same analysis.

$$\pi^{-1}(L) = \mathbb{P}\left\langle \begin{matrix} \text{spans of mons in } Y_i\text{'s} \\ \text{except pure } Y_0, Y_1 \end{matrix} \right\rangle \;\big\|$$

$\underline{\text{Concl}}$: All fibers of π are isomorphic to $\mathbb{P}^n$ for a fixed n

($Z \longrightarrow Gr(2,4)$ is a
"projective space bundle")

Dim Z = dim $Gr(2,4)$
+ dim fiber.

$= 4 +$

Fiber $= \mathbb{P} \langle$ mons in X_0, X_1, X_2, X_3 of deg d
except purely $X_0, X_1 \rangle$

$$\binom{d+3}{3} - (d+1)$$

fiber dim $= \binom{d+3}{3} - d - 2$

dim Z $= \binom{d+3}{3} - d + 2$

$$\begin{tikzcd} Z \arrow[d, "\phi"'] & \subset & \mathbb{P}V_d \times Gr(2,4) \\ \mathbb{P}V_d & & \end{tikzcd}$$

irred of dim $\binom{d+3}{3} - d + 2$

$\mathbb{P}V_d$: dim $\binom{d+3}{3} - 1$

$Z \downarrow$

<u>Compare</u>: $\binom{d+3}{3} - 1$ vs $\binom{d+3}{3} - d + 2$

-1 vs $2 - d$

$d = 1, 2$: $<$

$d = 3$: $=$

$d = 4, \dots$: $>$

$d=4$:

$$\dim \mathbb{P}V_d > \dim Z$$

$\Rightarrow \phi(Z) \subsetneq \mathbb{P}V_d$

$\Rightarrow$ Thm: A general surface of deg 4 has no lines on it.

$\hookrightarrow$ i.e. $\exists$ Zariski open $U \subset \mathbb{P}V_d$ s.t. $\forall$ $F \in U$ the surface $V(F)$ has no lines on it.

$\dim \phi(Z)$ turns out to be one less.

Cor: There is a poly. cond on the coeffs of a deg 4 poly that tells whether there is a line on the surface.

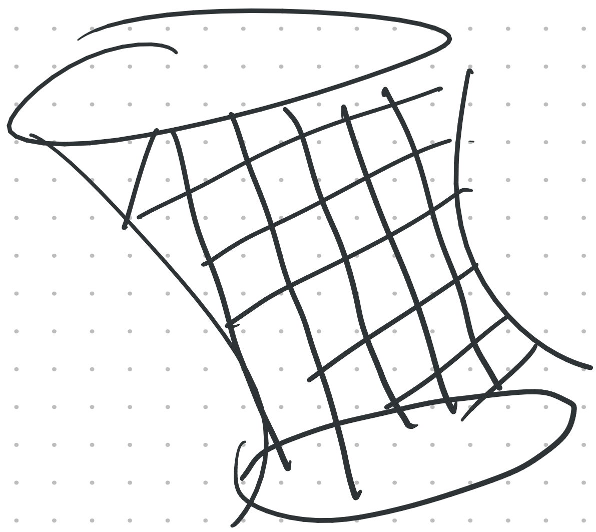
$\underline{d=2}$: $\dim Z = \dim \mathbb{P}V_d + 1$

$$\begin{array}{c} Z \\ \downarrow \\ \mathbb{P}V_d \end{array}$$

$\swarrow$ $\underline{\underline{\text{expect}}}$ that the map be surj & fiber dim $= \underline{\underline{1}}$

$\mathbb{P}^1 \times \{\cdot\}$

$\underline{\text{Quadric}} \cong \mathbb{P}^1 \times \mathbb{P}^1$



Deg 3 $\dim Z = \dim \mathbb{P}V_d$

$$\begin{array}{c} Z \\ \downarrow \\ \mathbb{P}V_d \end{array}$$

Expect map surj ✓
& fiber dim 0.

$\Downarrow$

fibers are finite

A generic cubic surface contains finitely many lines.

↳ Turns out this number is 27.

$\mathbb{P}^1 \times \mathbb{A}^2$

$\cup$

$$\left\{ \begin{matrix} X^2 - sY^2 = 0 \\ sX + tY = 0 \end{matrix} \right\} \xrightarrow{\pi_1} \mathbb{A}^2$$

$(0,1) \notin \mathrm{Im}(\pi)$

$$\left.\begin{matrix} s=0 \\ t=1 \end{matrix}\right\} \cancel{\begin{matrix} Y=0 \\ X=0 \end{matrix}}$$

$\exists$ open around $(0,1)$ disj from $\mathrm{Im}(\pi)$

$$\underbrace{X^2 = 0 \;\&\; Y = 0}_{\langle X^2, Y\rangle \supset \langle X, Y\rangle^3} \quad \text{at } (0,1)$$

✓

$(X^2 - sY^2) \cdot (aX + bY)$

$(sX + tY)(cX^2 + dXY + eY^2)$

$\longrightarrow$ Cubics

5 dim space a, b, c, d, e, f $\longrightarrow$ 4 dim space of cubics.

$$\begin{array}{c} X^3 \\ X^2Y \\ XY^2 \\ Y^3 \end{array} \begin{bmatrix} 1 & & & \\ & 1 & & \\ -s & & & \\ & -s & \cdots & \end{bmatrix}$$

rank 4 at $(s,t) = (0,1)$.

$\exists$ invertible $\underbrace{4 \times 4 \text{ minor}}$.

open condition.