$$\{\text{Proj var}/\mathbb{C}\} \xrightarrow{\pi} \left\{\begin{matrix}\text{Compact}\\ \text{top space}\end{matrix}\right\}$$

The image consists of "discrete" set of objects.

$$\left\{\begin{matrix}\text{Alg struct.}\\ \text{on } T\end{matrix}\right\} \longrightarrow T$$

↳ Param. by an algebraic Variety

<u>Example</u> ① Smooth curves

<u>Topology</u> :- genus <u>g</u>

$g=0$ $\quad$ $g=1$ $\quad$ $g=2,3,\ldots$

But what if $k \neq \mathbb{C}$?

A lot of this can be done purely algebraically.

genus g: Over $\mathbb{C}$, this g characterises the topology.

Many def. of g

$k = \overline{\mathbb{F}}_p$

X sm proj curve.

① # holes

② $2-2g = \chi_{top}(X)$

$= \operatorname{rk} \underbrace{H^0(X,\mathbb{Z})}_{} - \operatorname{rk} H^1 + \operatorname{rk} H^2$

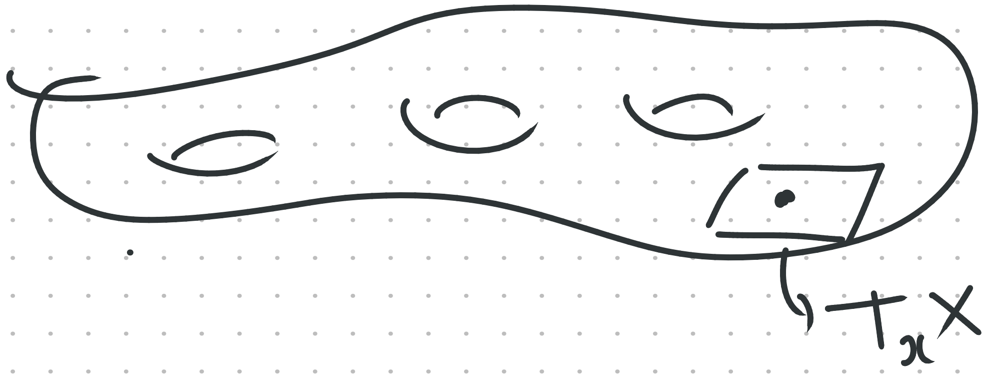
↳ uses topology.

↳ It is possible to define "topological" coh gps $H^1(X,\mathbb{C})$

"Etale cohomology"

③ Complex analytic def of g. $\boxed{k=\mathbb{C}}$

X : complex mfold of $\mathbb{C}$-dim 1



$\Omega(X)$:= Vector space of holomorphic differential forms on X

$X \ni x \rightsquigarrow T_x^*X$ = Cotangent space

ω diff form

$\omega : x \longmapsto \omega(x) \in T_x^*X$

A hol. form is ω "that varies holomorphically"

$\Omega(X)$ is a $\mathbb{C}$-vector space
↳ finite dim

Thm $\dim \Omega(X) = g$

X smooth proj curve / k
connected

Algebraic diff form

$\omega : X \ni x \longmapsto \omega(x) \in T_x^* X$

"varying algebraically"

$\Omega^{alg}(X)$ is k-vector space
finite dim.

If $k = \mathbb{C}$ then
$$\Omega^{alg}(X) = \Omega^{hol}(X)$$

(GAGA)

Def: The genus of X over any k to be $\dim_k \Omega^{alg}(X)$.

hol functions on $\mathbb{C} = \mathbb{A}^1$

exp, any conv. power series.

Alg $\subsetneq$ Hol.

genus of $\mathbb{P}^1_k = 0$

genus of $(Y^2Z = X^3 + aXZ^2 + bZ^3)$ is 1.

etc

$f(x)\,dx$

$g(y)\,dy$

$y = \frac{1}{x}$

$dy = -\frac{1}{x^2}\,dx$

Comparison thm. :-

$C \subset \mathbb{P}^2$ defined over $\underline{\mathbb{Z}}$.

$$C = V(Y^2Z - X^3 - XZ^2 - 3Z^3)$$

$\rightsquigarrow$ Avatar in $\mathbb{P}^2_{\mathbb{C}}$

$\mathbb{P}^2_{\mathbb{F}_p}$

$\mathbb{P}^2_k$ for any $\underline{k}$!

genus of C over $\mathbb{C}$ 

$=$ genus of C over k

↳ defined using $\underline{\Omega}^{alg}(C)$

Any field k

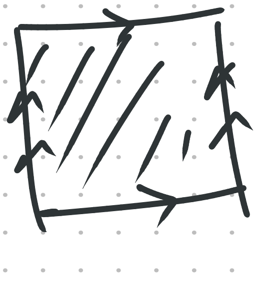
$\left\{ \begin{array}{c} \text{Sm. proj curves} \\ \text{of genus } g \end{array} \right\}$

↳ param. by an alg. var / k of dim $3g-3$. M_g.

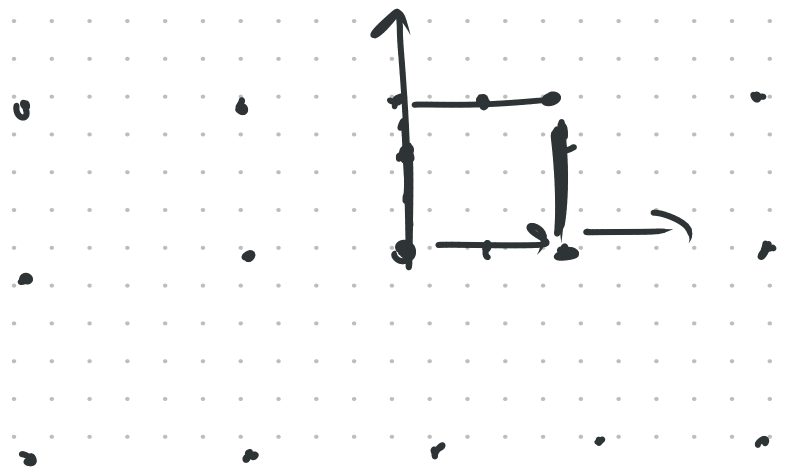
$g=1$:- $k = \mathbb{C}$

C smooth proj genus 1

C =  ← Group!



$T = \mathbb{R}^2/\mathbb{Z}^2$



$\mathbb{R}^2$ is a group + (abelian)

$\cup$

$\mathbb{Z}^2$ is a subgp

$\left(\mathbb{R}^2/\mathbb{Z}^2\right)$ is a gp.

C smooth proj curve / $\mathbb{C}$

↳ Has a group law!

Thm: This gp law is algebraic!

$$\left.\begin{array}{l} m: C \times C \longrightarrow C \\ i: C \longrightarrow C \end{array}\right\} \text{regular}$$

i.e. C is a group variety.

Can we describe the grp law algebraically? Without topology.

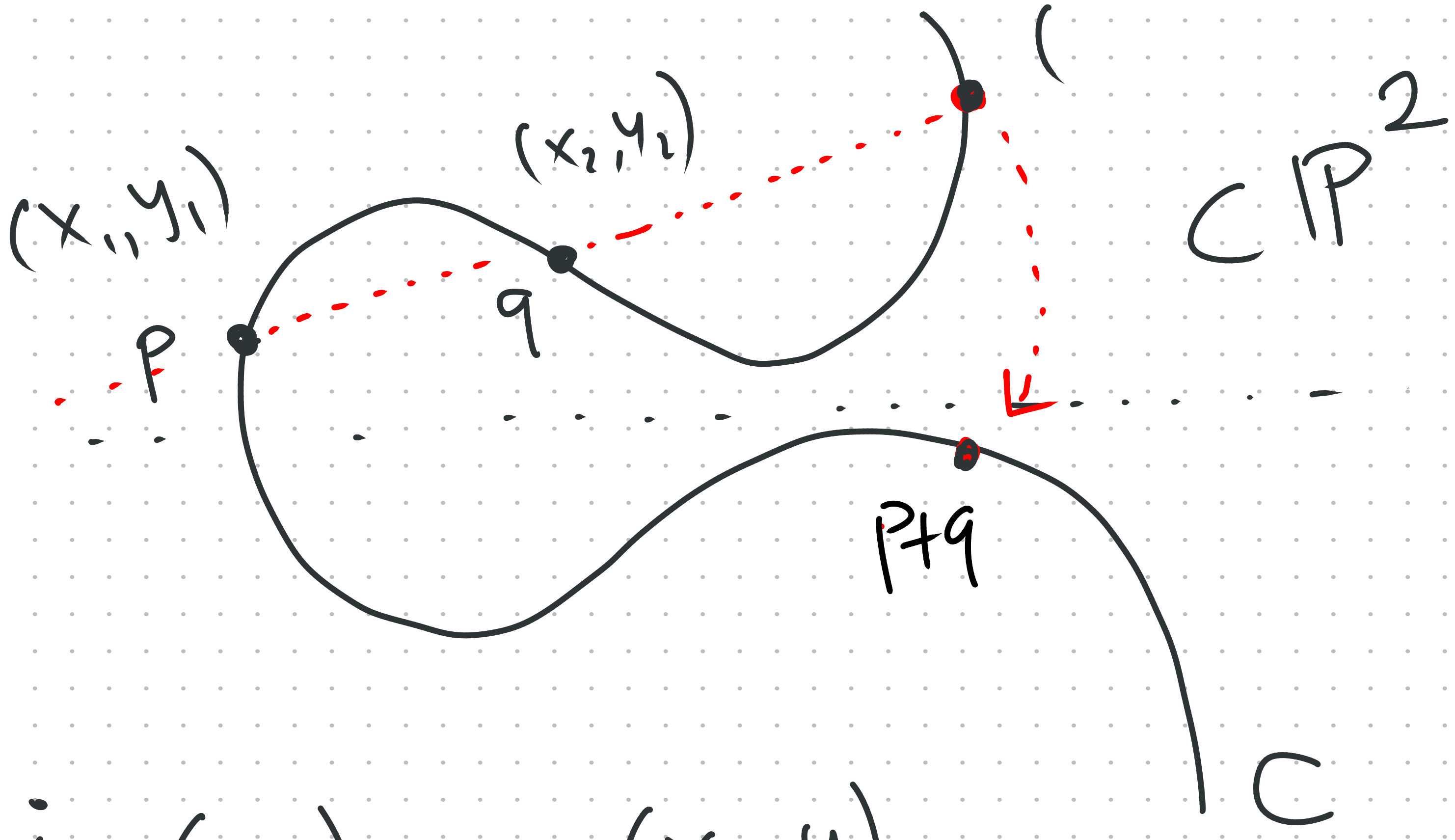
We can :-

Use $C =$

$$V(\text{cubic}) \subset \mathbb{P}^2$$

$$y^2 = x^3 + ax + b \quad (\text{projectivize})$$

$(x, y) \quad (x, -y)$



$i : (x,y) \mapsto (x,-y)$

↳ algebraic!

$p+q = m(p,q)$ ↙ as above

This procedure is algebraic.

& makes sense over any k

& actually defines a gp law.

& $k = \mathbb{C}$:– agrees with gp law on a torus.

Over k

Genus 1 curves $\leftarrow$ group varieties

over $\mathbb{C}$

$\boxed{\mathbb{R}^2/\mathbb{Z}^2}$

C/k $\leftarrow$ what group is it?

$C/\overline{\mathbb{F}_p}$?

C/k is a gp var. over k even if k is not alg. closed.

$C(k)$ $\leftarrow$ group what is it?

$C(\mathbb{F}_p) = ??$ gp

$C(\mathbb{F}_{p^2}) = ??$ gp

$\vdots$

$C(\mathbb{F}_q) = ??$ gp

Similar things hold in higher dim.n

$\mathbb{R}^4/\mathbb{Z}^4$ ∃ proj var look like

Abelian var of dim 2.

$\{ \cdots \}$

Abelian Var. of dim n $\boxed{\mathbb{R}^{2n}/\mathbb{Z}^{2n}}$