# THE CORRESPONDENCE THEOREMS FOR HOMOMORPHISMS AND SUBGROUPS

Let $G$ be a group, and $K \lhd G$ a normal subgroup. Let $\pi\colon G \to G/K$ be the homomorphism $g \mapsto [g]$. You may find it helpful to visualize this setup as in Figure 1. The goal of the correspondence theorems is the following.

**Goal.** Describe objects associated to $G/K$ (like homomorphism or subgroups) in terms of objects associated to $G$ and $K$.

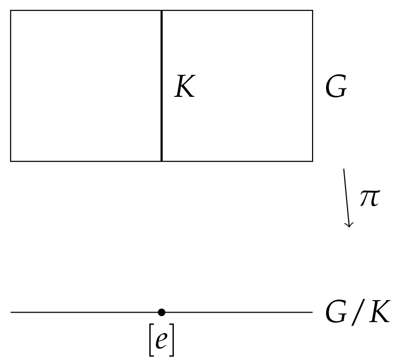


FIGURE 1. The setup of the correspondence theorems

By the first isomorphism theorem, if we have a surjective homomorphism $\phi\colon G \to G'$ and we let $K = \ker \phi$, then $G' \cong G/K$. So, via the first isomorphism theorem, our setup is equivalent to the setup of a surjective homomorphism $\phi\colon G \to G'$.

## 1. CORRESPONDENCE BETWEEN HOMOMORPHISMS

Let $H$ be a group and $\psi\colon G/K \to H$ a homomorphism. Let $\alpha = \psi \circ \pi$. Then $\alpha\colon G \to H$ is a homomorphism. Notice that $\alpha$ has the property that it maps $K$ to $\{e\}$. Indeed, for any $k \in K$, we have $\alpha(k) = \psi([k]) = \psi([e]) = e$. We thus get a correspondence

$$\left\{\begin{array}{l}\text{Homomorphisms}\\ \text{from } G/K \text{ to } H\end{array}\right\} \to \left\{\begin{array}{l}\text{Homomorphisms from } G \text{ to } H\\ \text{that map } K \text{ to } \{e\}\end{array}\right\}$$
$$\psi \quad \mapsto \quad \alpha = \psi \circ \pi.$$

**Theorem 1.** *Let $G$ be a group, $K \lhd G$ a normal subgroup and $\pi\colon G \to G/K$ the homomorphism $g \mapsto [g]$. Then we have a bijective correspondence*

$$\left\{\begin{array}{l}\textit{Homomorphisms}\\ \textit{from } G/K \textit{ to } H\end{array}\right\} \to \left\{\begin{array}{l}\textit{Homomorphisms from } G \textit{ to } H \textit{ that}\\ \textit{map } K \textit{ to } \{e\}\end{array}\right\}$$

*defined by*

$$\psi \quad \mapsto \quad \psi \circ \pi.$$

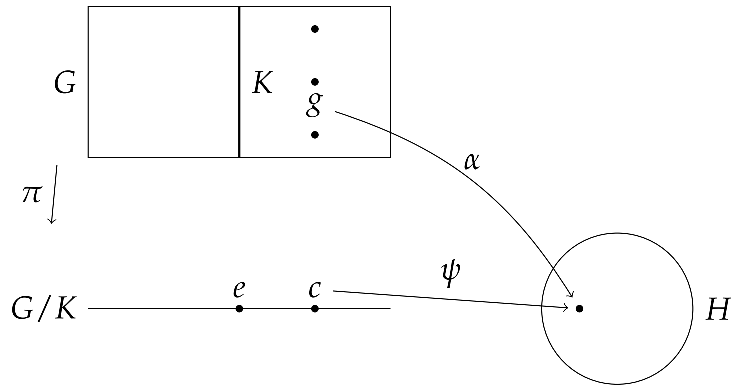


FIGURE 2. The construction of $\psi$ from $\alpha$

*Proof.* First, we show that the correspondence is injective. Suppose $\psi_1\colon G/K \to H$ and $\psi_2\colon G/K \to H$ are two homomorphisms such that $\psi_1 \circ \pi = \psi_2 \circ \pi$. Then we get

$$\psi_1(\pi(g)) = \psi_1([g]) = \psi_2([g]) = \psi_2(\pi(g))$$

for all $g \in G$. Since all elements of $G/K$ are of the form $[g]$, we get $\psi_1 = \psi_2$.

Second, we show that the correspondence is surjective. For this, given a homomorphism $\alpha\colon G \to H$ that sends $K$ to $\{e\}$, we must construct a homomorphism $\psi\colon G/K \to H$ such that $\alpha = \psi \circ \pi$. To construct $\psi$, we must specify $\psi(c)$ for every element $c$ of $G/K$. Recall that elements of $G/K$ are cosets of $K$ in $G$. Define

$$\psi(c) = \alpha(g) \text{ for } c = [g]. \tag{1}$$

See Figure 2 for a pictorial description of our definition. We must ensure that our definition of $\psi(c)$ does not depend on the representative $g$ chosen for $c$. To this end, suppose $[g_1] = [g_2] = c$. Then $g_1 = g_2 k$ for some $k \in K$. Since $\alpha$ maps $K$ to $\{e\}$, we get

$$\alpha(g_1) = \alpha(g_2 k) = \alpha(g_2)\alpha(k) = \alpha(g_2).$$

Thus, the value of $\psi(c)$ does not depend on which representative we choose. Therefore, (1) gives a well-defined function $\psi\colon G/K \to H$.

Having constructed $\psi$, checking that it is a homomorphism is straightforward. Indeed, by the definition of $\psi$ in (1), we have

$$\psi([g]) = \alpha(g).$$

Hence, we get

$$\psi([g_1][g_2]) = \psi([g_1 g_2]) = \alpha(g_1 g_2) = \alpha(g_1)\alpha(g_2) = \psi([g_1])\psi([g_2]).$$

Finally, saying $\psi([g]) = \alpha(g)$ is the same as saying $\psi(\pi(g)) = \alpha(g)$, which in turn, is the same as saying $\alpha = \psi \circ \pi$. Therefore, the correspondence is surjective as desired. $\square$

**Example.** Let us understand the correspondence theorem in a specific instance, namely for $\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} = \mathbf{Z}_2$. By the theorem, we have a one-to-one correspondence

$$\{\text{Homorphisms } \mathbf{Z}_2 \to G\} \leftrightarrow \{\text{Homomorphisms } \mathbf{Z} \to G \text{ that send } 2\mathbf{Z} \text{ to } \{e\}\}.$$

How do we go from left to right? Let a homomorphism $\psi\colon \mathbf{Z}_2 \to G$ be given. Then we get $\alpha\colon \mathbf{Z} \to G$ by setting $\alpha(i) = \psi([i])$.

How do we go from right to left? Let a homomorphism $\alpha\colon \mathbf{Z} \to G$ be given. Suppose $g = \alpha(1)$. Then $\alpha(i) = g^i$ because $\alpha$ is a homomorphism. See that $\alpha$ takes $2\mathbf{Z}$ to $\{e\}$ if and only if $\alpha(2) = g^2 = e$. Then we get $\psi\colon \mathbf{Z}_2 \to G$ by setting $\psi([i]) = g^i$. The choice of the representative $i$ does not matter: two representatives will differ by an even integer and $g$ raised to an even integer is $e$, so the answer will not change. Thus, we get a well-defined $\psi\colon \mathbf{Z}_2 \to G$.

Thus, the homomorphisms from $\mathbf{Z}_2$ to a group $G$ are in one-to-one correspendence with elements $g \in G$ with $g^2 = e$. For example, all homomorphisms from $\mathbf{Z}_2 \to S_4$ are given by $[i] \mapsto g^i$ where $g$ is one of the following:

$$\mathrm{id}, (12), (13), (14), (23), (24), (34), (12)(34), (13)(24), (14)(23).$$

So there are 10 homomorphisms from $\mathbf{Z}_2$ to $S_4$.

## 2. Correspondence between subgroups

This material is covered beautifully in the book (section §2.10). So use these notes as a supplement and not a replacement to that section. The setup in the book is that of a surjective homomorphism $G \to G'$. Our setup is that of a quotient by a normal subgroup $K$, namely $G \to G/K$. By the first isomorphism theorem, these are two equivalent situations.

Let $\overline{H} \subset G/K$ be a subgroup. Define $H \subset G$ by

$$H = \pi^{-1}(\overline{H}) = \{g \in G \mid [g] \in \overline{H}\}.$$

It is easy to check (check it!) that $H$ is a subgroup of $G$. Moreover, since $[k] = [e]$ for all $k \in K$ and $[e] \in \overline{H}$, the subgroup $H$ contains $K$. We thus get a correspondence

$$\begin{aligned} \{\text{Subgroups of } G/K\} &\to \{\text{Subgroups of } G \text{ that contain } K\} \\ \overline{H} &\mapsto H = \pi^{-1}(\overline{H}). \end{aligned} \tag{LR}$$

Let us construct a correspondence in the opposite direction. Let $H \subset G$ be a subgroup containing $K$. Define $\overline{H} \subset G/K$ by

$$\overline{H} = \pi(H).$$

Then $\overline{H}$ is a subgroup of $G/K$. Indeed, it the image of the group $H$ under the homomorphism $\pi$. We thus get a correspondence

$$\begin{aligned} \{\text{Subgroups of } G \text{ that contain } K\} &\to \{\text{Subgroups of } G/K\} \\ H &\mapsto \overline{H} = \pi(H). \end{aligned} \tag{RL}$$

**Theorem 2.** *Let $K \lhd G$ be a normal subgroup and $\pi\colon G \to G/K$ the standard homomorphism $g \mapsto [g]$. We have a bijective correspondence*

$$\{\textit{Subgroups of } G/K\} \leftrightarrow \{\textit{Subgroups of } G \textit{ that contain } K\} \tag{2}$$

*where the left to right direction is given by* (LR) *and the right to left direction is given by* (RL).

We will use the following lemma in the proof of the theorem. The proof of this lemma is a simple exercise with sets and functions. It has nothing to do with group theory. So I am leaving it to you.

**Lemma 3.** *Let $f\colon A \to B$ be a function between two sets. For a subset $T \subset B$ denote by $f^{-1}(T)$ the subset of $A$ given by*

$$f^{-1}(T) = \{a \in A \mid f(a) \in T\}.$$

*For a subset $S \subset A$, denote by $f(S)$ the subset of $B$ given by*

$$f(S) = \{f(s) \mid s \in S\}.$$

*Then $f(f^{-1}(T)) \subset T$ and $S \subset f^{-1}(f(S))$. Moreover, if $f$ is surjective, then $f(f^{-1}(T)) = T$.*

*Proof of the theorem.* Since we already have functions to and fro, what remains is to check that their compositions in both directions equal the identity.

Let us start with an $\overline{H}$ on the left. By applying (LR) to $\overline{H}$, we get $H = \pi^{-1}(\overline{H})$. We must show that applying (RL) to $H$ takes us back to $\overline{H}$, that is $\pi(H) = \overline{H}$. Since $\pi$ is surjective, this follows from Lemma 3.

Now, let us start with an $H$ on the right. By applying (RL) to $H$, we get $\overline{H} = \pi(H)$. We must show that applying (LR) to $\overline{H}$ takes us back to $H$, that is $\pi^{-1}(\overline{H}) = H$. The inclusion $H \subset \pi^{-1}(\overline{H})$ follows from Lemma 3. For the opposite inclusion, consider an element $g \in \pi^{-1}(\overline{H})$. Then $\pi(g) = [g] \in \overline{H}$. Since $\overline{H} = \pi(H)$, there is an $h \in H$ such that $[g] = [h]$. But $[g] = [h]$ means $g = hk$ for some $k \in K$. Since $H$ contains $K$, we get that $k \in H$. Therefore $g = hk \in H$. We thus get $\pi^{-1}(\overline{H}) \subset H$. Together with the opposite inclusion, we get $\pi^{-1}(\overline{H}) = H$. ☐

*Remark* 4. Under the correspondence of subgroups, normal subgroups of $G/K$ correspond to normal subgroups of $G$. This follows from Artin's Proposition 2.10.4 applied to the surjective homomorphism $\pi\colon G \to G/K$.