Modern Algebra 1: Midterm 1

October 2, 2013

- Answer the questions in the space provided.
- Give concise but adequate reasoning unless asked otherwise.
- You may use any statement from class, textbook, or homework without proof, but you must clearly write the statements you use.
- The exam contains 6 questions.
- At the end, there are some blank pages for scratch work. You may detatch them.

Name: ______________________________

Question	Points	Score
1	10	
2	8	
3	8	
4	8	
5	8	
6	8	
Total:	50	

1. (a) (4 points) State the definition of (i) a homomorphism (ii) the kernel of a homomorphism.

> **Solution:** A homomorphism from a group G to a group H is a function $\phi\colon G \to H$ such that $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$.
> The kernel of a homomorphism ϕ is defined by
> $$\ker\phi = \{g \in G \mid \phi(g) = e\}.$$

(b) (3 points) Give examples of two non-isomorphic finite groups of the same order. State in one sentence why they are not isomorphic.

> **Solution:** $\mathbf{Z}_6$ and $\mathbf{S}_3$ are both of order 6. They are not isomorphic because $\mathbf{Z}_6$ is abelian but $\mathbf{S}_3$ is not.

(c) (3 points) Find a homomorphism $\phi\colon \mathbf{Z}^+ \to \mathbf{S}_3$ whose kernel is $\mathbf{Z} \cdot 3$.
No justification is needed.

> **Solution:** Let $p = (123)$. Define ϕ by $\phi(n) = p^n$.

2. (a) (4 points) Express the permutation (1234) as a product of transpositions.

Solution: We have
$$(1234) = (14)(13)(12).$$

(b) (4 points) Find the sign of the permutation $(1234)(56)(78)$. Justify your answer.

Solution: We use that sgn is a homomorphism and $\operatorname{sgn}(\tau) = -1$ for all transpositions τ. We get

$$\begin{aligned}\operatorname{sgn}((1234)(56)(78)) &= \operatorname{sgn}((14)(13)(12)(56)(78)) \\ &= \operatorname{sgn}((14))\operatorname{sgn}((13))\operatorname{sgn}((12))\operatorname{sgn}((56))\operatorname{sgn}((78)) \\ &= (-1)^5 \\ &= -1\end{aligned}$$

3. The following is a partially filled multiplication table for a group. The element in row i and column j is $i * j$, where $*$ is the group operation.

$*$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	1	5		4
3	3	1	2	6		5
4	4	6	5			2
5	5					3
6	6	5	4	3	2	

(a) (4 points) What must be the values of $6 * 6$ and $5 * 2$? Give reasons.

Solution: Note that 1 is the identity. Since 6 must have an inverse, and none of $2, \ldots, 5$ is its inverse, we must have $6 * 6 = 1$.
Since $5 = 2 * 4$, we get

$$\begin{aligned} 5 * 2 &= (2 * 4) * 2 \\ &= 2 * (4 * 2) \text{ by associativity} \\ &= 2 * 6 \\ &= 4 \end{aligned}$$

(b) (4 points) Find a subgroup of order 2 and a subgroup of order 3.
No justification is necessary.

Solution: Subgroup of order 2: $\{1, 6\}$.
Subgroup of order 3: $\{1, 2, 3\}$.

4. (8 points) Prove that a group G is cyclic if and only if there exists a surjective homomorphism $\phi\colon \mathbf{Z}^+ \to G$.

Solution: Suppose G is cyclic. Let $x \in G$ be a generator. Then

$$G = \langle x \rangle = \{x^n \mid n \in \mathbf{Z}\}.$$

Define $\phi\colon \mathbf{Z}^+ \to G$ by $\phi(n) = x^n$. Then

$$\phi(m+n) = x^{m+n} = x^m x^n = \phi(m)\phi(n).$$

So ϕ is a homomorphism. Since

$$\operatorname{im}(\phi) = \{\phi(n) \mid n \in \mathbf{Z}\} = \{x^n \mid n \in \mathbf{Z}\} = G,$$

we see that ϕ is surjective.

Conversely, let $\phi\colon \mathbf{Z}^+ \to G$ be a surjective homomorphism. Let $x = \phi(1)$. Since ϕ is a homomorphism, we have $\phi(n) = x^n$ for all $n \in \mathbf{Z}$. So, we get

$$\operatorname{im}(\phi) = \{\phi(n) \mid n \in \mathbf{Z}\} = \{x^n \mid n \in \mathbf{Z}\} = \langle x \rangle.$$

Since ϕ is surjective, we have $\operatorname{im}(\phi) = G$. Therefore $G = \langle x \rangle$ is cyclic.

5. Which of the following are subgroups of $\mathrm{GL}_2(\mathbf{R})$? Justify your answer.

(a) (4 points) $G = \{M \in \mathrm{GL}_2(\mathbf{R}) \mid \det M > 0\}$

Solution: G is a subgroup. We check the three conditions.

1. Since $\det(I_2) = 1 > 0$, we have $I_2 \in G$.
2. If $A, B \in G$, then $\det(A) > 0$ and $\det(B) > 0$. Then $\det(AB) = \det(A)\det(B) > 0$. Hence $AB \in G$.
3. If $A \in G$, then $\det(A) = 0$. Then $\det(A^{-1}) = \det(A)^{-1} > 0$. Hence $A^{-1} \in G$.

(b) (4 points) $H = \{M \in \mathrm{GL}_2(\mathbf{R}) \mid M = M^{-1}\}$

Solution: H is *not* a subgroup—it is not closed under multiplication. Consider $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $A = A^{-1}$ and $B = B^{-1}$, so both A and B are in H. But $AB = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $(AB)^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so $AB \neq (AB)^{-1}$. Therefore, AB is not in H.

6. (8 points) Let G and H be two groups, $x \in G$ an element of order m and $y \in H$ an element of order n. Find, with proof, the order of (x, y) in $G \times H$ in terms of m and n.

Solution: We claim that the order of (x, y) is $\operatorname{lcm}(m, n)$.

For the proof, notice that $(x, y)^d = (e_G, e_H)$ if and only if $x^d = e_G$ and $y^d = e_H$. Since the order of x is m, we know that $x^d = e_G$ if and only if m divides d. Similarly, $y^d = e_H$ if and only if n divides d. Therefore,

$$(x, y)^d = (e_G, e_H) \text{ if and only if both } m \text{ and } n \text{ divide } d. \tag{1}$$

We know that the order of (x, y) is the the smallest positive integer d such that $(x, y)^d = (e_G, e_H)$. By (1), the order of (x, y) is the smallest positive integer multiple of m and n, which by definition is $\operatorname{lcm}(m, n)$.

Scratch Work

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