Modern Algebra 1: Midterm 2

November 11, 2013

- Answer the questions in the space provided.
- There are 5 questions. There is an additional bonus question at the end. Attempt it only if you have enough time.
- Give concise but adequate reasoning. You may use any statement from class or textbook without proof, but you must clearly state what you are using.
- At the end, there are some blank pages for scratch work. You may detach them.

Name: ____________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (a) (4 points) State the definition of a normal subgroup.

Solution: A subgroup $H \subset G$ is called a *normal subgroup* if $gHg^{-1} = H$ for all $g \in G$.
Equivalently, a subgroup $H \subset G$ is called a *normal subgroup* if $gH = Hg$ for all $g \in G$.

(b) (3 points) Give an example of a normal subgroup of S_4 other than $\{e\}$ or S_4. Explain why your example is a normal subgroup.

Solution: Consider the alternating group A_4 consisting of permutations in S_4 with sign $+1$. Then A_4 is a normal subgroup of S_4 because it is the kernel of the homomorphism $\mathrm{sgn} : S_4 \to \{\pm 1\}$.
Also, the set $\{\mathrm{id}, (12)(34), (14)(23), (13)(24)\}$ is a normal subgroup of S_4, being the kernel of a homomorphism $S_4 \to S_3$.

(c) (3 points) Give an example of a subgroup of S_4 that is not a normal subgroup. Explain why your example is not a normal subgroup.

Solution: Consider the two element subgroup $H = \{\mathrm{id}, (12)\}$. Taking $g = (13)$, we get $g(12)g^{-1} = (23) \notin H$. So H is not a normal subgroup.

2. (10 points) Let G be the subgroup of $\mathrm{GL}_2(\mathbf{R})$ defined by

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbf{R},\ ac \neq 0 \right\}.$$

Let $H \subset G$ be the subgroup defined by $a = c = 1$. Prove that H is a normal subgroup of G and identify G/H.

Solution: Define a function $\phi\colon G \to \mathbf{R}^\times \times \mathbf{R}^\times$ by

$$\phi \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = (a, c).$$

Since the matrix entries a and c can be any nonzero real numbers, ϕ is surjective.

Let us check that ϕ is a homomorphism. Let

$$M_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}, \text{ and } M_2 = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}.$$

Then

$$M_1 M_2 = \begin{pmatrix} a_1 a_2 & a_1 b_2 + b_1 c_2 \\ 0 & c_1 c_2 \end{pmatrix}.$$

Therefore, we get

$$\phi(M_1 M_2) = (a_1 a_2, c_1 c_2) = \phi(M_1)\phi(M_2).$$

Hence ϕ is a homomorphism.

Also, $\phi \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = (1, 1)$ if and only if $a = c = 1$. So, $\ker \phi = H$.

Since the kernel of a homomorphism is a normal subgroup, we deduce that H is a normal subgroup of G.

By the first isomorphism theorem, we get

$$G/H = G/\ker \phi \cong \operatorname{im} \phi = \mathbf{R}^\times \times \mathbf{R}^\times.$$

3. (10 points) Let G and H be finite groups whose orders are relatively prime (that is, $\gcd(|G|, |H|) = 1$). Show that the only homomorphism $\phi\colon G \to H$ is the trivial homomorphism: $\phi(g) = e$ for all $g \in G$.

Solution: Let $\phi\colon G \to H$ be a homomorphism. Then $\operatorname{im}\phi$ is a subgroup of H. By Lagrange's theorem, $|\operatorname{im}\phi|$ divides $|H|$.

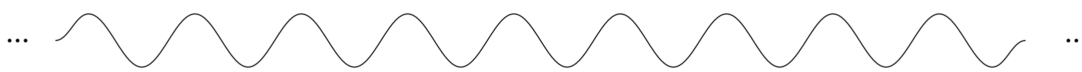
By the first isomorphism theorem, we have

$$G/\ker\phi \cong \operatorname{im}\phi.$$

In particular, $|G| = |\ker\phi||\operatorname{im}\phi|$. So, $|\operatorname{im}\phi|$ also divides $|G|$.

Since $\gcd(|G|, |H|) = 1$, and $|\operatorname{im}\phi|$ divides both $|G|$ and $|H|$, we conclude that $|\operatorname{im}\phi| = 1$. Since $e \in \operatorname{im}\phi$, we must have $\operatorname{im}\phi = \{e\}$. Therefore $\phi(g) = e$ for all $g \in G$.

4. Let G be the group of isometries of the infinite pattern



(a) (5 points) Find the point group of G.

Solution: Recall that the point group $\overline{G}$ of G is the image of G under the homomorphism

$$t_a A \mapsto A$$

from the group of all isometries to the group O_2 of isometries fixing the origin.

Observe that G contains a reflection, namely the reflection through the vertical line through any crest or trough. Therefore $\overline{G}$ contains the reflection in the Y-axis.

Note that G contains a rotation by π (about the midpoint between a crest and a trough). Hence $\overline{G}$ contains the rotation by π about the origin.

It is clear that G cannot contain a rotation by a (positive) angle smaller than π. From what we proved in class, $\overline{G}$ is generated by the reflection in the Y-axis and rotation by π. This group is D_2, given by

$$\overline{G} = D_2 = \{\mathrm{id}, r_x, r_y, \rho_\pi\},$$

where r_x is the rotation in the x-axis, r_y is the rotation in the Y-axis, and ρ_π the rotation by π about the origin.

We can also see directly that $\overline{G}$ contains the reflection in the X-axis by observing that G contains a glide along the X-axis.

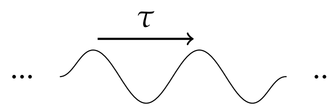
Alternatively, we can get $\overline{G}$ without using the above statement from class as follows. Suppose $t_a A$ is an isometry of the pattern, where $A \in O_2$. Note that $t_a A$ must send the X-axis to the X-axis. Since t_a sends a line to a parallel line, A must send the X-axis to a horizontal line. But A preserves the origin. So A must send the X-axis to itself. By orthogonality, A must send the Y-axis to the Y-axis. Since A is orthogonal and preserves the two axes, it can only be one of

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now it is easy to check that all four possibilities are present (they come from the identity, a vertical reflection, a horizontal glide, and a rotation by π of the original pattern). Thus,

$$\overline{G} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

(b) (5 points) Let τ be the translation by one wave-length.



Find the number of subgroups of G containing τ.

Solution: Any subgroup of G containing τ must contain the group $\langle \tau \rangle$ generated by τ. By the definition of the point group $\overline{G}$, we have a surjective homomorphism

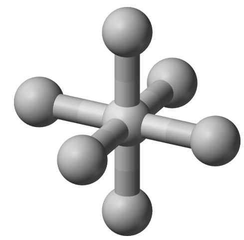
$$\phi \colon G \to \overline{G},$$

whose kernel consists of the translations in G. But the translations in G are precisely the elements of $\langle \tau \rangle$. So we get

$$\overline{G} \cong G / \ker \phi = G / \langle \tau \rangle.$$

By the correspondence theorem for subgroups, the subgroups of G containing $\langle \tau \rangle$ are in bijection with the subgroups of $\overline{G}$. But $\overline{G}$ is isomorphic to the Klein four group, which has 5 subgroups: $\{\mathrm{id}\}$, $\{\mathrm{id}, r_x\}$, $\{\mathrm{id}, r_y\}$, $\{\mathrm{id}, \rho_\pi\}$, and $\{\mathrm{id}, r_x, r_y, \rho_\pi\}$. Hence there are five subgroups of G containing τ.

5. Let G be the group of orientation preserving isometries of a molecule of SF_6 (sulfur hexafluoride). In coordinates, the central S atom is $(0,0,0)$ and the six F atoms are $(\pm 1,0,0)$, $(0,\pm 1,0)$ and $(0,0,\pm 1)$.



(a) (5 points) Find the order of G.

Solution: Consider the action of G on the set of F atoms. All F atoms form one orbit. The stabilizer of an F atom contains four elements, namely the four rotations about the line joining that atom to S by angles 0, $\pi/2$, π and $3\pi/2$. Remember that since we are only considering orientation preserving isometries, we must not count reflections.

By the orbit-stabilizer formula, we get

$$|G| = |O_F||G_F| = 6 \cdot 4 = 24.$$

(b) (5 points) Show that there is a surjective homomorphism $G \to S_3$.

Solution: Let $S = \{X, Y, Z\}$ be the set of the three coordinate axes. See that any isometry in G must take an axis to another axis. We thus get an action of G on S. Since S contains three elements, such an action gives a homomorphism

$$\phi\colon G \to S_3.$$

We now check that ϕ is surjective. Consider the element $g \in G$ which is the rotation by $\pi/2$ about the positive Z-axis. Then $\phi(g)$ fixes the Z axis, but switches the X and Y axes. In other words, $\phi(g) = (XY)$. Similarly, by taking h which is the rotation by $\pi/2$ about the positive Y axis, we get $\phi(h) = (XZ)$. Therefore, both (XY) and (XZ) are in $\operatorname{im}\phi$. Since any permutation of X, Y, Z can be written as a product of (XY) and (XZ), and $\operatorname{im}\phi$ is closed under products, we get $\operatorname{im}\phi = S_3$. That is, ϕ is surjective.

(c) (3 points (bonus)) Identify G.

Solution: $G \cong S_4$.

To see why, we first find a homomorphism $G \to S_4$. Such a homomorphism is equivalent to an action of G on a set with four elements. What set-of-four can we see in the picture? We have 8 octants, given by the 8 possible sign patterns of X, Y, and Z, namely $(+,+,+)$, $(+,+,-)$, etc, and we see that G must act on the set of octants. But 8 is too many—we want 4.

Now we see that if an isometry sends an octant O to an octant O', then it must send the octant opposite to O to the octant opposite to O' (the opposite octant is obtained by switching all three signs). We can thus pair the 8 octants into 4 pairs of opposite octants. Setting

$$S = \{\text{Pairs of opposite octants}\},$$

we get an action of G on S, and thus a homomorphism

$$\phi\colon G \to S_4.$$

Since both sides have the same number of elements, either surjectivity or injectivity of ϕ implies that it is an isomorphism. I'll leave it to you to check this.

Scratch Work

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