# MODERN ALGEBRA 1: HOMEWORK 1

(1) For sets $A$, $B$, and $C$, prove that
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$
Illustrate this with a Venn diagram.

(2) For a set $S$, the *power set of* $S$ is the set of all subsets of $S$. In symbols,
$$P(S) = \{A \mid A \subset S\}.$$
  (a) Write down the power set of $\{0, 1\}$.
  (b) Write down the power set of $\emptyset$.
  (c) Let $S$ be a finite set with $|S| = n$. What is the cardinality of $P(S)$? Justify your answer.

(3) Let $S = \{1, 2, 3\}$. Find two specific functions $f\colon S \to S$ and $g\colon S \to S$ such that $f \circ g \neq g \circ f$. Do the same with $S$ replaced by the real numbers $\mathbf{R}$.

(4) We saw in class that if both $f$ and $g$ are injective then $g \circ f$ is injective.
  (a) Prove that if $g \circ f$ is injective, then $f$ is injective.
  (b) Give an example to show that if $g \circ f$ is injective, then $g$ need not be injective.

(5) Formulate and solve the correct version of the previous problem (both parts) with "injective" replaced by "surjective."

(6) Let $A$ be a finite set and $f\colon A \to A$ a function. Show that $f$ is injective if and only if $f$ is surjective.
Show that the above statement is not true for infinite sets. In other words, give an example of an infinite set $A$ and a function $f\colon A \to A$ which is injective but not surjective and a function $g\colon A \to A$ which is surjective but not injective.

(7) Let $f\colon A \to B$ be a map of sets. A map $g\colon B \to A$ is called an *inverse* of $f$ if $g \circ f\colon A \to A$ is the identity map of $A$ and $f \circ g\colon B \to B$ is the identity map of $B$. Show that $f$ admits an inverse if and only if $f$ is bijective.
(The identity map $\mathrm{id} : S \to S$ is the map that sends $x$ to $x$ for all $x \in S$.)

(8) Let $\phi\colon \mathbf{Z} \to \mathbf{R}$ be the map defined by
$$\phi(n) = n^3 - 3n + 1.$$
Is $\phi$ surjective? Is $\phi$ injective? Justify your answers.